MA580/CSC580 Review II

1. Homework problems; problems from class; class notes etc.

2. Perform Gauss elimination with partial column pivoting on the matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ 2 & 4 & 2 \end{bmatrix}.$$

   (a) What is the LU factorization of PA (where $PA = LU$)?

   (b) Compute the direct factorization $A = LU$.

   (c) Approximately how many operations are required in (a) and (b).

   (d) Compute the determinant of A from (a) and (b).

   (e) Use your factorization results from (a) and (b) to solve $Ax = b$, where $b^T = \begin{bmatrix} 2 & -4 & 6 \end{bmatrix}$.

   (f) Find $\|A\|_p$, $cond_p(A)$ for $p = 1, 2, \infty$.

3. When we solve a linear system of $Ax = b$ ($det(A) \neq 0$) on a computer with machine epsilon ϵ using the Gaussian elimination with partial column pivoting (GEPP), the final computed solution, say x_c will be different from the true solution $x_e = A^{-1}b$. **(a):** Give an error estimate of the relative error. **(b):** Give an upper bound of the growth factor $g(n)$. **(c):** Let the residual of x_c be $r(x_c) = \|b - Ax_c\|$, show that $\dfrac{\|\mathbf{x}_e - \mathbf{x}_a\|}{\|\mathbf{x}_c\|} \leq \|A^{-1}\| \dfrac{\|r(\mathbf{x}_c)\|}{\|\mathbf{x}_c\|}$.

4. List at least three kinds of matrices for which pivoting strategies may not be necessary and explain why.

5. Suppose

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1/5 & 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) Can L_1 or L_3 be a Gauss transformation matrix with partial pivoting? Why?

   (b) Compute L_1^{-1}, L_3^{-1}, $L_1 L_3$, and $L_1^{-1} L_3^{-1}$.

   (c) Compute P^{-1}, P^T, P^2, PL_3, and PL_3P.

   (d) Find the condition number of each matrix.

6. For the following matrices

$$A = \begin{bmatrix} 3 & -1 & \alpha \\ -1 & \beta & 1/2 \\ 1 & 1/2 & \gamma \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 0 & 1 \\ 1 & \alpha & -1 & 1 \\ 0 & -1 & \beta & \gamma \\ 1 & 1 & 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & \gamma & -2 \\ \beta & 2 & \gamma \\ -2 & 5 & 4 \end{bmatrix},$$

can you choose the parameters so that the matrix is

(a) symmetric;
(b) strictly row diagonally dominant;
(c) symmetric positive definite?

7. Suppose $A = LDL^T$, where L is a unit lower triangular matrix.

(a) Is A symmetric?
(b) When is A a symmetric positive definite matrix?
(c) Show that such a decomposition is possible if and only if the determinants of the principal leading sub-matrices A_k of A are all non-zero for $k = 1, 2, \cdots n-1$.
(d) What are the orders of operations (multiplication/division, addition/subtraction) needed for such decomposition?
(e) Can you get $A = LL^T$ factorization from $A = LDL^T$ if A is a S.P.D? How?

8. Derive $A = LU$ decomposition, where U **is a unit upper triangular matrix**. That is to derive the recursive relation for

$$\begin{aligned} &l_{ij}, \qquad i = j,\, j+1,\, \cdots,\, n, \\ &u_{ij}, \qquad j = i+1,\, \cdots,\, n, \end{aligned}$$

(a) Write a pseudo code for your algorithm.
(b) How many operations (multiplications/divisions and addition/subtractions) are required in your algorithm.
(c) Outline how to use such a decomposition to solve $Ax = b$ and compute the determinant of A.

9. Give a vector of $\bar{x}$, and $Ax = b$. Derive the relation between the residual $\|b - A\bar{x}\|$ and the error $\|\bar{x} - A^{-1}b\|$.

10. For the following model matrices, what kind of matrix-factorization would you like to use for solving the linear system of equations? Analyze your choices (operation count, storage, pivoting etc).

$$\begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0.01 & 3 & 0 & -4 \\ 1 & 2 & 1 & 2 \\ -1 & 0 & 3 & -2 \\ 5 & -2 & 3 & 6 \end{bmatrix}.$$

11. Let $A^{(2)}$ is the matrix obtained after one step Gauss elimination applied to a matrix A, that is

$$a_{ij}^{(2)} = a_{ij} - \frac{a_{i1}}{a_{11}}\, a_{1j}.$$

   (a) Show that

$$\max_{ij} |a_{ij}^{(2)}| \;\le\; 2 \max_{ij} |a_{ij}| \qquad (1)$$

   if partial pivoting is used.

   (b) Show that without pivoting, (1) is still true if A is column diagonally dominant.

12. If we use Gauss-Seidel iteration backwards, that is, from $x_n, x_n - 1, \cdots x_1$, we get another Gaus-Seidel iterative method. Derive the matrix and vector form of such an iterative method. If we apply this iterative method and the Gauss-Seidel method alternatively, then the combined method is called symmetric Gauss-Seidel method. The related SOR(ω) is called symmetric SOR(ω) or SSOR(ω) . Write a pseudo-code.

13. (a): Show that if λ_i is an eigenvalue of A, then λ^k is an eigenvalue of A^k for any integers k assuming that if $k < 0$ A^{-1} exists. (b): Show that if $A = A^T$, then $cond_2(A) = \dfrac{\max|\lambda_i(A)|}{\min|\lambda_i(A)|}$.

14. If λ is an eigenvalue of a matrix A,

   (a) show that $|\lambda| \le ||A||$ and so $\rho(A) \le ||A||$;

   (b) Show that if there is one matrix norms such that $\|R\| < 1$, then the stationary iterative method converges.

   (c) In the SOR(ω) method, can we take $\omega = -0.5, 1.8, 3.4$?

15. Given a stationary iterative method $x^{k+1} = Rx^k + c$, show that **(a):** if there is one matrix norm such that $\|R\| < 1$, then the iterative method converges. **(b):** If the spectral radius $\rho(R) > 1$, then the iterative method diverges.

16. Check the convergence of Jacobi, Gauss-Seidel, and SOR(ω) if (a): A is a column dominant matrix (stricly, weakly) ; (b): A is a symmetric positive (or negative) matrix.

17. Consider the linear system of equations

$$\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}}{h^2} = f_{ij}, \quad 0 < i,\, j < n-1,$$

   with zero boundary condition ($i = 0$, or $j = 0$, or $i = m$, or $j = n$. (a): What is the size of the linear system of equations if it is written as $AU = F$? (b): What are the order of $\|A\|_p$, $cond_2(A)$ in terms of n? (c): Show that $\rho(R_J) = \max\limits_{1\le k\le n} \left| 1 + \dfrac{\lambda_k(A)}{4} \right|$. (d): * Show that the eigenvalues of A are

$$\lambda_{i,j} = -\left(4 - 2\left(\cos\frac{i\pi}{n} + \cos\frac{j\pi}{n}\right)\right), \quad i, j = 1, 2, \cdots n-1.$$

   (d): How many iterations do we need in general if we use Jacobi, Gauss-Seidel, and SOR(ω) in terms of n?

18. Consider the linear system of equations (also consider one, three dimensions)

$$U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij} + \alpha U_{ij} = f_{ij}, \quad 0 < i,\, j < n-1. \qquad (2)$$

   (a) Write down the Jacobi, SOR(ω) methods.

   (b) What is the order of the number of iterations for the convergence when $\omega = 1$, and the optimal ω in terms of n when $\alpha = 0$?

   (c) When $\alpha = 0$ write down the matrix vector form of the linear system of equations using the natural ordering, how about black/red ordering. What is relation between the two coefficient matrices. Does SOR(ω) converge? For which ω, why?

19. Given the following linear system of equations:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 3 \\ 2x_2 + x_3 &= 2 \\ -x_2 + 2x_3 &= 2 \end{aligned}$$

   (a) With $x^{(0)} = [1, -1, 1]^T$, find the *first* and *second* iteration of the Jacobi, Gauss-Seidel, and SOR ($\omega = 1.5$) methods.

   (b) Write down the Jacobi and Gauss-Seidel iteration matrices R_J and R_{GS}.

   (c) Do the Jacobi and Gauss-Seidel iterative methods converge?