MA/CSC 580, Homework #7
Suggested

1. Given a linear system of equations $Ax = b$, where A is an n by n symmetric positive definite matrix.

   (a) Show that we can find a matrix B which is also an SPD matrix such that $B^2 = A$. Thus we can denote $B = A^{1/2}$.

   (b) Show that x^* that minimizes $\phi x = \frac{1}{2}x^T Ax - b^T x$ is the solution $x^* = A^{-1}b$ and vice-versa.

   (c) Show that $x^T Ax$ is a vector norm, denoted as $\|x\|_{A^{1/2}}$.

   (d) Show that $\sqrt{\lambda_{min}(A)}\,\|x\|_2 \le \|x\|_{A^{1/2}} \le \sqrt{\lambda_{max}(A)}\,\|x\|_2$.

   (e) Given an initial guess x_0 and $r(x_0) \neq 0$. Let $p_0 = r_0$. Consider the conjugate method. For $k = 0, 1, \cdots$ until converge

   $$\begin{aligned} x_{k+1} &= x_k + \alpha_k p_k, \qquad \text{thus} \quad r_{k+1} = r_k - \alpha_k A p_k, \\ p_{k+1} &= r_{k+1} + \beta_{k+1} p_k \end{aligned}$$

   Define $e_k = A^{-1}b - x_k$. Find α_k to minimize $\min_{\alpha_k} \|e_{k+1}\|_{A^{1/2}}$.

   (f) Find β_{k+1} such that $p_{k+1}^T A p_k = 0$.

   (g) Show that $x_{k+1} - x_0$, p_k, r_k, are all in the Krylov space $\mathcal{K}_k(r_0)$.

   (h) Show that $\{A^{1/2}p_j,\ j = 0, 1, \cdots, k-1\|$ form an orthogonal basis in $\mathcal{K}_k(r_0)$ in 2-norm, i.e. regular Euclidian norm.

   (i) Show that $\|e_k\|_{A^{1/2}} \le min_{p \in P_k, p(0)=1} \|p(A)\|_2 \, \|e_0\|_{A^{1/2}}$. Where P_k is a set of all polynomials of with degree $\le k$.

   (j) Show that the CG method convergences in at most $\min\{p_{min}(A), n\}$ steps, where $p_{min}(A)$ is the degree of the minimal of polynomial of A.

   (k) $r_k^T p_j = 0$, $j = 1, 2, \cdots k-1$.