MA/CSC 580, Homework #6
Due November 30, 2018

1. Let $w \neq 0 \in R^n$, $A = w^T w$, $B = ww^T$.
   (a) Find the size of the matrices A and B.
   (b) Find $\|A\|_p$ and $\|B\|_p$, $p = 1, 2, \infty$.
   (c) Find the rank of A and B.
   (d) Find $cond_p(A)$ and $cond_p(B)$ if they are meaningful.
   (e) Find all eigenvalues and corresponding eigenvectors of A and B.
   (f) Find the peudoinverse of A and B.

2. Let $A \in R^{n,n}$ be a diagonalizable matrix, E is a perturbed matrix satisfying $\|E\| \leq C\epsilon$. **(a)**: show that

$$\min_i \left| \lambda_i(A) - \lambda_i(A+E) \right| \leq cond(S)\|E\|,$$

where S is a matrix such that $S^{-1}AS = D = diag(\lambda_1(A), \lambda_2(A), \cdots \lambda_n(A))$. What conclusion can you get if $A = A^T$?

Hint: Consider $(A+E)x = \lambda_i(A+E)x$, and thus $(\lambda_i(A+E) - A)x = Ex$.

(b): If A and B are the following matrices, what are the eigenvalues for ech matrix? Find perturbation bounds for each individual eigenvalue.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

3.
   (a) Show that the least squares solution of

$$\begin{bmatrix} 0_1 \\ B \\ 0_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   is $x^* = B^{-1}b_2$, where 0_1 and 0_2 are zero matrices, and B is an invertible square matrix.

   (b) Use the result you proved in Part a to solve the following least squares problems and calculate the 2-norm of the residual vector for each solution you computed.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}; \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix}$$

4. Use the QR method to solve the over-determined system $Ax = b$, where

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix}.$$

(a) $b = \begin{bmatrix} 3 & 0 & 0 & 6 & -8 \end{bmatrix}^T$.

(b) $b = e_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$.

(c) Use the Given's rotation to find the least squares solution when $b = e_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$. **Hint:** Use the result in previous problem to determine which element that you wish to put as zero.

Calculate the 2-norm of the residual vector for each solution you computed.

Hint: You can use Matlab for intermediate computation and verification. The solution in Matlab can be written as $x_s = pinv(A) * b$ which may not be the same as $x_{ls} = A \setminus b$.

5. **(a):** Find the singular value decomposition for the matrix (with the process)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b): Use your result to solve the linear system of equations $Ax = b$, where $b = [2,\ 1]^T$. **(c):** Explain the meaning of the solution.

6. Given data: $y_i = \sin(2t_i) + 0.5 * \cos(5t_i) + \epsilon\,\mathrm{rand}((t(i)),\ i = 0, 1, \cdots, m.\ t_i = 1 + i(\pi - 1)/m$. Find the least squares solution using the following models. **Hint:** $M = 10; t = 1 : (pi - 1)/M : pi$; $y = sin(2 * t) + 0.5 * cos(5 * t) + 0.01. * rand(size(t))$;

(a) $y(t) \sim \displaystyle\sum_{i=0}^{n} a_i t^i$.

(b) $y(t) \sim \displaystyle\sum_{i=0}^{int(n/2)} (a_i \cos(i * t) + b_i \sin(i * t))$.

(c) $y(t) \sim a_0\, e^{a_1 t}$.

Take $M = 10$; $n = 2$ and $n = 10$, $\epsilon = 0$, and $\epsilon = 0.1$. Plot your approximations over the interval $(1, \pi)$ using 100 points $x_p = 1 : (pi - 1)/100 : pi$. Compare the true solution and approximated solutions when $\epsilon = 0$.