MA/CSC 580, Homework #5
Due Tuesday, November 16, 2018

1. Let $A \in R^{n,n}$ be a square matrix.

   (a) Let $(\lambda, \mathbf{x})$ be an eigenpair of A. Show that $(\lambda^k, \mathbf{x})$ is an eigenpair of A^k for any integer k. Can k be negative or zero?

   (b) If $A = A^H$ (the conjugate transpose of A and so on) be a symmetric matrix, use the conclusion above to show the following

   $$\|A\|_2 = \max_{1\le i\le n} |\lambda_i(A)|, \qquad \|A^{-1}\|_2 = \frac{1}{\min_{1\le i\le n}|\lambda_i(A)|}, \qquad cond_2(A) = \frac{\max_{1\le i\le n}|\lambda_i(A)|}{\min_{1\le i\le n}|\lambda_i(A)|}.$$

   (c) Also show that all eigenvalues of A are real and eigenvectors corresponding to different eigenvalues are orthogonal. **Hint:** Consider $Ax = \lambda_i x$, and $y^H A^H = \bar{\lambda}_i y$, $y^h Ax$ and $x^h Ay$.

   (d) If A is a lower/upper triangular matrix, show that $\lambda_i(A) = a_{ii}$, that is, the eigenvalues of A are the diagonal entries. Is such a matrix always diagonalizable?

2. Let $A \in R^{n,n}$ be the following matrix

   $$A = \begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_n \end{pmatrix},$$

   Assume that $\lambda_i \neq \lambda_j$ if $i \neq j$. Find a similarity transform so that the Gershgorin circles of the transformed matrix are separated. **Hint:** See the proof of the convergence of stationery iterative methods.

3. (a): Derive the Power method using $\|\mathbf{x}\|_1$ or $\|\mathbf{x}\|_\infty$ scaling and show its convergence under appropriate conditions. (b): Compare with different method for general and symmetric matrices. You can write a Matlab code to test different methods; and then carry out some analysis.

4.

   (a) Use the Gershgorin's theorem to locate the intervals that contain the eigenvalues of A

   $$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -6 & 0 \\ 0 & 1 & 9 \end{bmatrix}.$$

   (b) Can A have complex eigenvalues? Why?

   (c) Is A diagonalizable? Why?

   (d) Apply one step Power method using the 2-norm, and the x_p notation.

   (e) Assume that eigenvalues A satisfy $|\lambda_1| > |\lambda_2| > |\lambda_3|$, apply one step ***shifted inverse Power method*** to approximate the eigenvalue λ_2 and its eigenvector with initial guess $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

   Hint: You can use Matlb command [l u p]= lu(A) to find $PA = LU$ decomposition.

5. Let $x = \begin{bmatrix} 3 & 0 & -1 & 2 \end{bmatrix}^T$, and $y = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^T$.

   (a) Is there a Householder matrix P such that $Px = y$? Explain.

   (b) Let $\tilde{y} = \alpha y$, find the scalar α and a Householder matrix P such that $Px = \tilde{y}$.

   (c) Find the QR decomposition of the following matrix.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

6. Write a computer program to find the least dominant eigenvalue of the matrix A and corresponding the unit eigenvector. Test your code for the following matrices:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 2 & -1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix};$$

   C is a n by n random matrix generated by (i), generate a diagonal matrix, $n = 10; D = diag(rand(n, 1) * 100); S = rand(n, n); C = inv(S) * D * S$. In this way, we can generate a matrix with known eigenvalues.

   Tabulate the number of iterations, the relative error of the eigenvalue, and the residue vector $\|Ax - \lambda x\|_2$ for $n = 10, 20, 40, 80, 160$. For the second matrix, also try $n = 11, 21, 41, 81, 161$. Explain your results. **Hint:** Use the Lemma learned in class to find the exact eigenvalues of A for the first matrix; and the Matlab function $eig(A)$ for the second matrix.

7. **Extra credit:** Use the scaled column pivoting method to solve/verify Problem 4 in Homework 3. The Matlab code cond_hw.m can be found the course web-page and in Doodle.