Due, October 31, 2018

1. Let A be a symmetric positive definite matrix, so it has the Cholesky decomposition $A = LL^T$. Show that (a): $0 < l_{kk} \le \sqrt{a_{kk}}$, $k = 1, 2, \cdots, n$. (b): From (a) to derive $\max_{1\le j\le i\le n} |l_{ij}| \le \sqrt{\max_{1\le i,j\le n} |a_{ij}|}$. That is, the Cholesky decomposition is a stable algorithm. (c): Do the Jacobi, Gauss-Seidel, and SOR(ω) iterative methods converge?

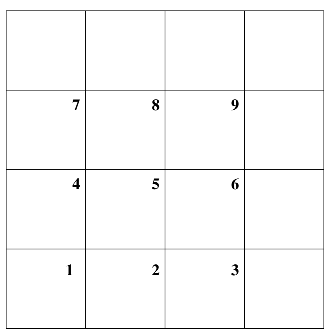
2. Consider the Poisson equation

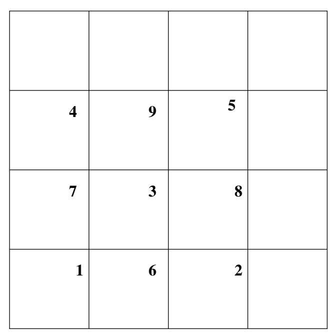
$$\begin{aligned} u_{xx} + u_{yy} &= xy, \quad (x, y) \in \Omega \\ u(x, y)|_{\partial\Omega} &= 0, \end{aligned}$$

where Ω is the unit square. Using the finite difference method, we can get a linear system of equations

$$\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}}{h^2} = f(x_i, y_j), \quad 1 \le i, j \le 3, \tag{1}$$

where $h = 1/4$, $x_i = ih$, $y_j = jh$, $i, j = 0, 1, 2, 3, 4$, and U_{ij} is an approximation of $u(x_i, y_j)$. Write down the coefficient matrix and the right hand side using the *red-black* orderings given in the right diagram below. What is the dimension of the coefficient matrix? How many nonzero entries and how many zeros? Generalize your results to general case when $0 \le i, j \le n$ and $h = 1/n$. Write down the component form of the $SOR(\omega)$ iterative method. Does the $SOR(\omega)$ iterative method depend on the ordering? From your analysis, explain whether you prefer to use Gaussian elimination method or an iterative method.





3. Given the following linear system of equations:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 3 \\ 2x_2 + x_3 &= 2 \\ -x_2 + 2x_3 &= 2 \end{aligned}$$

   (a) With $x^{(0)} = [1, -1, 1]^T$, find the *first* iteration of the Jacobi, Gauss-Seidel, and SOR ($\omega = 1.5$) methods.

   (b) Write down the Jacobi, Gauss-Seidel, and Seidel iteration matrices R_J , R_{GS}, and SOR(ω).

   (c) Do the Jacobi and Gauss-Seidel iterative methods converge? Why?

4. Explain when we want to use *iterative* methods to solve linear system of equations $Ax = b$ instead of *direct* methods.

   Also if $||R|| = 1/10$, then the iterative method $x^{(k+1)} = R\,x^{(k)} + c$ converges to the solution x^*, $x^* = Rx^* + c$. How many iterations are required so that $||x^{(k)} - x^*|| \le 10^{-6}$? Suppose $||x^{(0)} - x^*|| = O(1)$.

5. Judge whether the iterative method $x^{(k+1)} = R\,x^{(k)} + c$ converges or not.

$$(a): \quad R = \begin{bmatrix} e^{-1} & -e^{1} & -1 & -1 & -10 \\ 0 & \sin\pi/4 & 10^4 & -1 & -1 \\ 0 & 0 & -0.1 & -1 & 1 \\ 0 & 0 & 0 & 1-e^{-2} & -1 \\ 0 & 0 & 0 & 0 & 1-\sin(\alpha\pi) \end{bmatrix}, \qquad (b): \quad \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.3 & -0.7 \\ 0 & 0.69 & 0.2999 \end{bmatrix}$$

6. Determine the convergence of the Jacobi and Gauss-Seidel method applied to the system of equations $Ax = b$, where

$$(a): \quad A = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}, \qquad (b): \quad A = \begin{bmatrix} 3 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ 2 & 3 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & 3 & -1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 2 & 3 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & 2 & 3 \end{bmatrix}$$

7. Modify the Matlab code *poisson_drive.m* and *poisson_sor.m* to solve the following diffusion and convection equation:

$$u_{xx} + u_{yy} + au_x - bu_y = f(x, y), \quad 0 \le x, y \le 1,$$

   Assume that solution at the boundary $x = 0$, $x = 1$, $y = 0$, $y = 1$ are given (Dirichlet boundary conditions). The central-upwinding finite difference scheme is

$$\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{ij}}{h^2} + a\frac{U_{i+1,j} - U_{ij}}{h} - b\frac{U_{i,j} - U_{i,j-1}}{h} = f_{ij}$$

   (a) Assume the exact solution is $u(x, y) = e^{2y}\sin(\pi x)$, find $f(x, y)$.

   (b) Use the $u(x, y)$ above for the boundary condition and the $f(x, y)$ above for the partial differential equation. Let $a = 1$, $b = 2$, and $a = 100$, $b = 2$, solve the problem with $n = 20$, 40, 80, and $n = 160$. Try $\omega = 1$, the best ω for the Poisson equation discussed in the class, the optimal ω by testing, for example $\omega = 1.9, 1.8, \cdots, 1$.

   (c) Tabulate the error, the number of iterations for $n = 20$, 40, 80, and $n = 160$ with your tested optimal ω, compare the number of iterations with the Gauss-Seidel method.

   (d) Plot the solution and the error for $n = 40$ with your tested optimal ω. Label your plots as well.

8. **Extra credit**. Do some research to explain the behavior using cond_hw.m and possible ways of improving it.