MA/CSC 580
Homework #2
Due Tuesday, September 26, 2018

1. Find $||x||_p,\ p = 1, 2, \infty$ for the following vectors
   (a) $\mathbf{x} = (3, -4, 0, -3/2)^T$.
   (b) $\mathbf{x} = (\sin k, \cos k, 2^k)^T$ for a fixed positive integer k.
   (c) $\mathbf{x} = (4/(k+1), -2/k^2, k^2 e^{-k})^T$ for a fixed positive integer k.

2. Find $||A||_p,\ p = 1, 2, \infty$ for the following matrices:
$$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}; \qquad \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix};$$

3. (a) Show that $\|x\|_\infty$ is equivalent to $\|x\|_2$. That is to find constants C and c such that $c \le \|x\|_\infty \le \|x\|_2 \le C\|x\|_\infty$. Note that you need to determine such constants that the equalities are true for some particular x.
   (b) Show that $\|Qx\|_2 = \|x\|_2$ f Q is an orthogonal matrix ($Q^H Q = I$, $QQ^H = I$).
   (c) Show that $\|AB\| \le \|A\|\|B\|$ for any natural matrix norm, and $\|QA\|_2 = \|A\|_2$.

4. Given
$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$$
   (a) Use Gaussian elimination with the partial pivoting to find the matrix decomposition $PA = LU$. This is a paper problem and you are asked to use exact calculations (use fractions if necessary).
   (b) Find the determinant of the matrix A.
   (c) Use the factorization to solve $Ax = b$.

5. Consider solving $AX = B$ for X, $A \in R^{n,n}$, $B \in R^{n,m}$. There are two obvious algorithms. The first one is to get $A = PLU$ using Gaussian elimination, and then to solve for each column of X by forward and backward substitution. The second algorithm is to compute A^{-1} using Gaussian elimination and then to multiply $A^{-1}B$ to get X. Count the number of operations by each algorithm and determine which one is faster.

6. (Programming Part) Given a sequence of data
$$(x_1, y_1),\ (x_2, y_2),\ \cdots,\ (x_m, y_m), (x_{m+1}, y_{m+1}),$$
write a program to interpolate the data using the following model
$$y(x) = a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} + a_m x^m.$$

(a) Derive the linear system of equations for the interpolation problem.

(b) Let $x_i = (i-1)h$, $i = 1, 2, \cdots, m+1$, $h = 1/m$, $y_i = \sin \pi x_i$, write a computer code using the Gaussian elimination with column partial pivoting to solve the problem. Test your code with $m = 4, 8, 16, 32, 64$ and plot the error $|y(x) - \sin \pi x|$ with 100 or more points between 0 and 1, that is, predict the function at more points in addition to the sample points. For example, you can set $h1 = 1/100$; $x1 = 0 : h1 : 1$, $\quad y1(i) = a_0 + a_1 x1(i) + \cdots + a_{m-1}(x1(i))^{m-1} + a_m(x1(i))^m$, $y2(i) = \sin(\pi x1(i))$, $plot(x1, y1 - y2)$.

(c) Record the CPU time (in Matlab type *help cputime*) for $m = 50, 100, 150, 200, \cdots, 350, 400$. Plot the CPU time versus m. Then use the Matlab function polyfit $z = polyfit(m, cputime(m), 3)$ to find a cubic fitting of the CPU time versus m. Write down the polynomial and analyze your result. Does it look like a cubic function?

7. **Extra Credit:** Choose **one** from the following (Note: please do not ask the instructor about the solution since it is extra credit):

(a) Let $A \in R^{n\times n}$. Show that $\|A\|_2 = \max_{1\le i\le n} \sqrt{\lambda_i(A^TA)}$ and $\|A^{-1}\|_2 = \dfrac{1}{\min_{1\le i\le n} \sqrt{\lambda_i(A^TA)}}$, where $\lambda_i(A^TA)$, $i = 1, 2, \cdots, n$ are the eigenvalues of A^TA. Show further that $cond_2(A) = \dfrac{\sigma_{max}}{\sigma_{min}}$, where $\sigma_{max}, \sigma_{min}$ are the largest and smallest nonzero singular values of A.

(b) Show that if A is a symmetric positive definite matrix, then after one step of Gaussian elimination (without pivoting), then reduced matrix A_1 in

$$A \Longrightarrow \begin{bmatrix} a_{11} & * \\ 0 & A_1 \end{bmatrix}$$

must be symmetric positive definite. Therefore no pivoting is necessary.