Is this Jacobi or Gauss-Seidel iteration? why

$\vec{x}^{(k+1)} := \vec{x}^{(k)}$ (X2=X1)

for $i=1:n$

$$x_i^{(k+1)} = \left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}\, x_j^{(k+1)}\right) \Big/ a_{ii}$$

0.2 end

How many $\vec{x}^{(k)}$ do we need to store? for J, G-S.

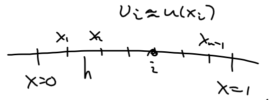
2   2. for stopping criteria.

$x^{(k)} := \vec{x}^{(k+1)}$

Oct 15-9:31 AM

Ex: ODE BVP $\begin{cases} u''(x) = f(x), & 0<x<1 \\ u(0)=0, \ u(1)=0 \end{cases}$

$U_i \approx u(x_i)$



Given an n, $h=\frac{1}{n}$

$x_i = ih$, $i=0,1,\dots,n$. $x_0=0$, $x_n=1$.

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i), \quad i=1,2,\dots,n-1.$$

$U_0=0$, $U_n=0$

$AU=F$, $A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}$. $\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{n-1}\end{bmatrix}$

Ordering: Index of unknown and equation

Using the same ordering.

Oct 15-9:40 AM

The coefficients in front of diagonals are $-\frac{2}{h^2}$. Solve for the diagonal

$$U_i = \frac{U_{i-1} + U_{i+1}}{2} + \frac{h^2}{2} f(x_i)$$

Form an iteration

$$U_i^{(k+1)} = \frac{U_{i-1}^{(k)} + U_{i+1}^{(k)}}{2} - \frac{h^2}{2} f(x_i), \quad i=1,2,\dots,n-1.$$

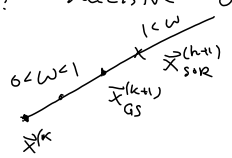
Jacobi iteration.

$\vec{U}^{(k+1)} := \vec{U}^{(k)}$

$$U_i^{(k+1)} = \frac{U_{i-1}^{(k+1)} + U_{i+1}^{(k+1)}}{2} - \frac{h^2}{2} f(x_i), \quad \text{G-S.}$$

Oct 15-9:48 AM

SOR(ω): Successive over-relaxation



$0<\omega<1$

$1<\omega<2$

$$\vec{x}_{SOR}^{(k+1)} = (1-\omega)x^{(k)} + \omega\, x_{GS}^{(k+1)}$$

$\omega=0$, $x^{(k)}$, $\omega=1$, $x_{GS}^{(k+1)}$

$0<\omega<1$, interpolation

$1<\omega<2$, extrapolation $\to$ over-relation.

Oct 15-9:54 AM

Component form. Implementation.   Matrix-vector convergence analysis

$$\to\ x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \omega\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right) \Big/ a_{ii}$$

$i=1,2,\dots,n.$

Pseudo-code

$\vec{x}^{(k+1)} := \vec{x}^{(k)}$

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \omega\left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^{(k+1)}\right) \Big/ a_{ii}$$

$i=1,2,\dots,n.$ ω is a given parameter.

How do we chose ω. $\omega \sim 2$.

Oct 15-10:02 AM

Matrix-vector form of SOR(ω)

$$\underline{(\quad)}\,\vec{x}^{(k+1)} = (\quad)\vec{x}^{(k)} + \vec{c}$$

$$\vec{x}^{(k+1)} = \underbrace{(\quad)^{-1}(\quad)}_{R}\vec{x}^{(k)} + (\quad)^{-1}\vec{c}$$

$$D\vec{x}^{(k+1)} - \omega L x^{(k+1)} = (1-\omega)D\vec{x}^{(k)} + \omega U\vec{x}^{(k)} + \omega\vec{b}$$

$A = D - L - U$

$$(D-\omega L)\vec{x}^{(k+1)} = \left((1-\omega)D + \omega U\right)\vec{x}^{(k)} + \omega\vec{b}$$

$$\vec{x}^{(k+1)} = \underline{(D-\omega L)^{-1}\left((1-\omega)D + \omega U\right)}\vec{x}^{(k)} + (D-\omega L)^{-1}\omega b$$

$$= R\vec{x}^{(k)} + \vec{c}$$

$R_{SOR(\omega)} = (D-\omega L)^{-1}\left((1-\omega)D + \omega U\right)$, $c = (D-\omega L)^{-1}\omega\vec{b}$.

If $\omega=1$, $R_{SOR(1)} = R_{GS} = (D-L)^{-1}U$

$c = (D-L)^{-1}\vec{b}$

Oct 15-10:08 AM

Is this the SOR iteration?

$$\vec{X}_{GS}^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right) / a_{ii}$$ ✓

$$\vec{X}_{SOR(\omega)} = (1-\omega)\vec{x}^{(k)} + \omega \vec{X}_{GS}^{(k+1)}$$ ✓

Does not use the most updated info.

Oct 15-10:17 AM

Convergence. Fix-point iteration.

$$\vec{x}^{(k+1)} = G(\vec{x}^{(k)}), \quad |G'| < 1$$

$$\vec{x}^{(k+1)} = \underline{R\vec{x}^{(k)} + c}$$

$$\frac{DG(R\vec{x}^{(k)} + c)}{} = R \quad \|R\| < 1$$

A sufficient condition

<u>Thm:</u> If there is a matrix norm such that $\|R\| < 1$ then the iterative method will converge.

$$\|\vec{x}^* - \vec{x}^{(k)}\| \le \frac{\|R\|^k}{1 - \|R\|} \|\vec{x}^{(1)} - \vec{x}^*\|$$

$$\vec{x}^* = R\vec{x}^* + c$$

$$\|R\|^k \to 0$$

Oct 15-10:21 AM

Convergence of basic iterative methods.

$$x^{k+1} = Rx^k$$

$$\vec{x}^{k+1} = R\,\vec{x}^k + c$$

Consistent condition $\underline{x^* = Rx^* + c}$ is solvable

$$\lim_{k\to\infty} \vec{x}^k = x^* \text{ for any } \vec{x}^0, \text{ initial}$$

<u>Thm:</u> If there is a matrix norm such that $\|R\| < 1$, then the iterative method always converges.

Oct 17-9:29 AM

Proof. $x^* = Rx^* + c$

$$x^{k+1} = Rx^k + c$$

$$x^{k+1} - x^* = R(x^k - x^*)$$
$$= R\,R\,(x^{k-1} - x^*)$$
$$= \cdots = R^{k+1}(x^0 - x^*) \checkmark$$

$$\|x^{k+1} - x^*\| \le \|R^k\|\,\|x^0 - x^*\|$$

$\|R\|^k$ $\qquad \|R\| < 1$

$$\|R^k = R^{k-1} \cdot R\| \le \|R^{k-1}\|\,\|R\|$$
$$\le \|R^{k-2}\|\,\|R\|\,\|R\| \le \|R\|^k$$

$$\lim_{k\to\infty} \|\vec{x}^{k+1} - \vec{x}^*\| = 0 \quad \text{Since } \|R\| < 1 \checkmark$$

Oct 17-9:52 AM

Convergence speed. If we know $\|R\| < 1$, then how many iterations do we need such that $\|e^k\| \le 10^{-8}$?

$$x^{k+1} = R\,x^k + c$$
$$x^k = R\,x^{k-1} + c$$

$x^{k+1} = Rx^k + c$

$x^* = Rx^* + c$

$$x^{k+1} - x^k = R(x^k - x^{k-1}) = R\,R\,(x^{k-1} - x^{k-2})$$
$$\underline{x^{k+1} - x^*} - (x^k - x^*) = R^k(x^1 - x^0)$$
$$R(x^k - x^*) - (x^k - x^*) = R^k(x^1 - x^0)$$
$$(R - I)(x^k - x^*) = R^k(x^1 - x^0)$$
$$-(x^k - x^*) = (I - R)^{-1} R^k (x^1 - x^0)$$
$$\|x^k - x^*\| \le \|(I - R)^{-1}\|\,\|R\|^k\,\|x^1 - x^0\|$$
$$\le \frac{\|R\|^k}{1 - \|R\|}\|x^1 - x^0\|$$

Convergence Speed

$$\|R\|^k \le 10^{-8}$$
$$k \log_{10}\|R\| \le -8$$
$$k \ge -8 / \log_{10}\|R\|$$

Oct 17-9:58 AM

Ex: $R = \begin{bmatrix} 0.9 & 0.05 \\ 0.8 & 0.1 \end{bmatrix}$ $\qquad \vec{x}^{k+1} = R\vec{x}^k + c$

Converges or not? $\|R\|_\infty = \max\{0.95, 0.9\} = 0.95 < 1$

$\|R\|_1 = 1.7$

Ex: $R = \begin{bmatrix} 0.2 & 0.81 \\ 0.81 & 0 \end{bmatrix}$ $\qquad \|R\|_1 = \|R\|_\infty = 1.01$

No conclusion.

Oct 17-10:09 AM

If and only if condition, is $\rho(R) < 1$.

The spectral radius is defined by

$$\rho(R) = \max_{1 \le i \le n} |\lambda_i(A)|$$

Sketch of the proof. $\underline{\rho(R) < 1} \to \|R\| < 1$

From linear algebra, there is a matrix S such that

$$S^{-1} R S = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_p \end{bmatrix} = J, \quad J_k = \begin{bmatrix} \lambda_k & 1 & & 0 \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{bmatrix}$$

One block $\|J\|_\infty = \rho(R)$, $\quad \|J\|_\infty = \begin{matrix}\rho(R) \\ \rho(R)+1\end{matrix}$

Otherwise $\le \rho(R) + 1$

Oct 17-10:12 AM

$$\underline{D^{-1}(\varepsilon) S^{-1} R S D} = \begin{bmatrix} J_1(\varepsilon) & & & 0 \\ & J_2(\varepsilon) & & \\ & & \ddots & \\ 0 & & & J_p(\varepsilon) \end{bmatrix}$$

$$J_k(\varepsilon) = \begin{bmatrix} \lambda_k & \varepsilon & & \\ & \lambda_k & \varepsilon & \\ & & \ddots & \varepsilon \\ 0 & & & \lambda_k \end{bmatrix}$$

$\rho(R) + \varepsilon < 1$

$\varepsilon = \dfrac{1 - \rho(R)}{2}$

$$D = \begin{bmatrix} D_1(\varepsilon) & & & 0 \\ & D_2(\varepsilon) & & \\ & & \ddots & \\ 0 & & & D_p(\varepsilon) \end{bmatrix}$$

$$D_k(\varepsilon) = \begin{bmatrix} 1 & & & 0 \\ & \varepsilon & & \\ & & \varepsilon^2 & \\ 0 & & & \varepsilon^{s-1} \end{bmatrix}$$

$D_k \in \mathbb{R}^{s,s}$

$$T^{-1} R T = \begin{bmatrix} J_1(\varepsilon) & & 0 \\ & \ddots & \\ 0 & & J_p(\varepsilon) \end{bmatrix} = \underline{J(\varepsilon)}$$

$$\underline{\|T^{-1} R T\|_\infty} \le \rho(R) + \varepsilon < 1$$

$T = SD$

$T^{-1} = D^{-1} S^{-1}$

Define a new matrix norm,

$\|B\| = \|T^{-1} B T\|_\infty$, can you prove it is a matrix norm.

Oct 17-10:19 AM

Convergence of $x^{k+1} = Rx^k + c$

If and only if condition $\rho(R) < 1$.

Sufficient: If $\rho(R) < 1$, $\longrightarrow$ convergence

A: New matrix norm

$$\|R\|_{new} = \|D^{-1} \underbrace{S^{-1} A S}\, D\|_\infty$$

B: In the Jordan blocks, we can use any non-zero number.

necessary condition: If it converges, then $\rho(R) < 1$. Counter proof

If not true, $\rho(R) \overset{=}{>} 1$. There is an eigen-pair (λ^*, x^*) such that

$$R x^* = \lambda^* x^* \qquad |\lambda^*| = \rho(R) > 1$$

$$x^0 = x^*, \quad x^1 = R x^0 = \lambda^* x^0 \qquad x^* = \vec{0}.$$

$$x^2 = R x^1 = \lambda^* R x^0 = (\lambda^*)^2 x^0$$

$$x^{k+1} = (\lambda^*)^{k+1} x^0 \longrightarrow \infty \qquad x^{k+1} = (-1)^1 x^k$$

$h = 1$

no limit.

Special matrices.

Type A: Strict row diagonally dominant

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad \text{SRDD} \qquad i=1,2,\ldots,n$$

<u>Thm.</u> If A is SRDD, then both Jacobi, G-S iterative methods converge.

$$A x = b \qquad x^{k+1} = \cancel{R} x^{k} + c$$

$$\rho(R_{G\text{-}S}) \overset{?}{\leq} \rho(R_J) < 1. \qquad A = D - L - U$$

$$R_J = D^{-1}(L+U) = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & \ddots & -\frac{a_{23}}{a_{22}} \cdots & \frac{a_{2n}}{a_{2n}} \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

$$\|R_J\| = \max_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} = \max_i \frac{\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|}{|a_{ii}|} < 1$$

Type B. S.P.D. symmetric positive definite.

Thm. If A is an S.P.D, then Gauss-Seidel and SOR(ω), $0<\omega<2$ converge (as well as Jacobi).

The proof is not easy.

Ex: $A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}$

$$\begin{cases} -u'' = f, & a < x < b \\ u(a) = \alpha, & u(b) = \beta \end{cases}$$

Is A SRDD?

$i \quad |+2| \geq |-1| + |-1| \qquad |2| > |-1|$

Weakly row diagonally dominant. $\qquad |2| > |-1|$

If A is "irreducible" and weakly row diagonally dominant, then Jacobi, G-S, SOR(ω), $0<\omega<2$ converge $\geq$

Weakly diagonally dominant

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \checkmark$$ and at least

one of them satisfies $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

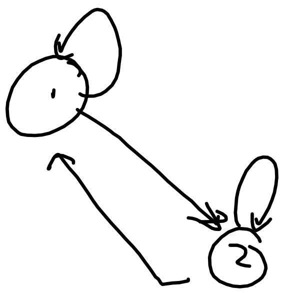
A is irreducible. means there is no permutation matrix $P = \underline{P_{12} P_{23}} \cdots P_{ij}$ such that

$$P^T A P = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad P_{ij}^{-1} = P_{ij}^T$$

P^{-1} $\quad Ax = b$

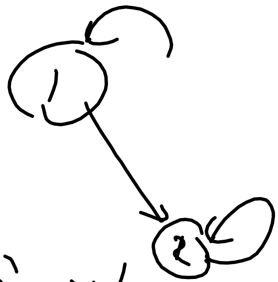
$$P^T A P P^T x = P^T b = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} P^T x = P^T b$$

Graph theory.



$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$ probality matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$



maximum principle $\longrightarrow$ elliptic, parabolic — not with wave eqn,

A is reducible.

For Poisson Eqn. convergence speed

best ω. $\rho(R)$

1D. $A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}$

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i)$$

SOR(ω)

$$U_i^{k+1} = \frac{U_{i-1}^k + U_{i+1}^k}{2} + \frac{h^2 f(x_i)}{2}, \quad i = 1, 2, \dots, n-1.$$

$$R_J = D^{-1}(L+U)$$

$\lambda_i(R)$, $\underline{\lambda_i(A)}$ known

$$\det(\lambda I - D^{-1}(L+U)) = 0$$

$$\det(D^{-1}(\lambda D - (L+U))) = 0$$

$A = D - L - U$

$$\cancel{\det(D^{-1})}\det(\lambda D - (L+U)) = 0$$

$\det(AB) = \det(A)\det(B)$

$$\det((\lambda - 1)D + D - (L+U)) = 0$$

$$\det((\lambda-1)D + A) = 0$$

$$\det(+2(\lambda-1)I + A) = 0$$

$-\det(\lambda_i I - A)$

$\underline{-2(1-\lambda)}I - A$

...

$$\lambda_J = \frac{-\lambda_i(A)}{2} + 1 \quad \rho(R_J).$$