SVD decomposition and pseudo-inverse

Q: Given $x \in \mathbb{R}^n$, $\|x\|_2 = 1$, can you generate an orthogonal basis in $\mathbb{R}^n$?

$Q = [x_1, x_2, \dots, x_n]$, $Q^H Q = Q^H Q = I$.

Solution:

$[e_1, e_2, \dots, e_n] = I$

Find a Householder matrix

$P x_1 = e_1$

$x_1 = P^T e_1$

$P^T = P^{-1} = P$

$Q = P [e_1, e_2, \dots, e_n]$

$Q = [x_1, Pe_2, \dots, Pe_n]$

$\det(P) = \pm 1$

$P^T P = P^2 = I$

For any $A \in \mathbb{C}^{m \times n}$,

$U^H U = I$

$$A = U \Sigma V^H$$

$U \in \mathbb{C}^{m \times m}$

$V \in \mathbb{C}^{n \times n}$

$\|A\|_2 = \sigma_1$

$$\Sigma = \begin{bmatrix} \begin{matrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_p \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}$$

$V^H V = I$

$\text{rank}(A) = p$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p > 0$$

$A^+ \in \mathbb{C}^{n \times m}$.

$A^+ = V \Sigma^+ U^H$ is called the pseudo-inverse of A.

$Ax = b$

$$\boxed{x = A^+ b}$$

Proof:

$$A^H A = V \Sigma^H \underline{U^H U} \Sigma V^H$$

$$= V \Sigma^H \Sigma V^H$$

$$= V \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ 0 & & \sigma_p^2 & 0 \\ & 0 & & 0 \end{bmatrix}_{n \times n} V^H$$

$\sigma_i^2 > 0$ are non-zero eigenvalues of $A^H A$ (AA^H)

There is a vector $x \neq 0$, $\|x\|_2 = 1$,

$\|Ax\|_2 = \sigma_1$

$A^H \underline{\underset{m \times n}{A} x} = \sigma_1^2 x$

x^H

Expand x_1 to form an orthogonal basis in R^n

$\|Ax\|_2^2 = \sigma_1^2$

$V = [x_1, V_1] \in R^{n \times n}$, $V_1 \in R^{n, n-1}$

$$\overset{U_1}{[\quad]} A \overset{\hat{V}_1}{[\quad]} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & A_1 \end{bmatrix}$$

x, v_1

$y A^H A x$

c

cx

Let $y_1 = \frac{A x_1}{\sigma_1}$, $\|y_1\|_2 = \frac{\|Ax_1\|}{\sigma_1}$
$= \sigma_1/\sigma_1 = 1$

Expand y_1 to form an orthogonal matrix in R^m

$A^H A x = \sigma_1^2 x.$

$U = [y_1, U_1]$, $U_1 \in R^{m, m-1}$

$$U^H A V = \begin{bmatrix} y_1^H \\ U_1^H \end{bmatrix} A [x_1, V_1]$$

$[Ax_1, AV_1]$

$$= \begin{bmatrix} y_1^H \underline{Ax_1} & \omega^T \\ 0 & A_1 \end{bmatrix}$$

$Ax_1 \parallel y_1$

$U_1 y_1 = 0$

$y_1^H U_1^H$

$$= \begin{bmatrix} y_1^H \sigma_1 y_1 & \omega^T \\ 0 & A_1 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \omega^T \\ 0 & A_1 \end{bmatrix}$$

The remaining proof is to show $\omega^T = 0$

$$\|U^H A V\|_2 = \left\| \begin{bmatrix} \sigma_1 & w^T \\ 0 & A_1 \end{bmatrix} \right\|_2 = \|A\|_2 = \sigma_1$$

$A^H A$ and $A A^H$ have the same non-zero eigenvalues.

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

$$\left\| \begin{bmatrix} \sigma_1 & w^T \\ 0 & A_1 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} \sigma_1 & 0 \\ w & A_1^H \end{bmatrix} \right\|_2$$

$$\geq \left\| \begin{bmatrix} \sigma_1 & 0 \\ w & A_1^H \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2$$

$$= \left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2 \geq \sqrt{\sigma_1^2 + \|w\|_2^2}$$

Only possible if $w = 0$.

$$\begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}$$

Ex: Let $A = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$, find the SVD decomposition. $\sigma_1 = 2$.

$$A^T A = [\sqrt{2} \ \ \sqrt{2}] \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = 4$$

$x_1 = [1]$. $V = [1]$

$$y_1 = \frac{Ax_1}{\sigma_1} = \frac{1}{2} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} (1) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[y_1 \ y_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & x \\ \frac{1}{\sqrt{2}} & x \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$U^T A V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} (1)$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$