Perturbation Theory for algebraic eigenvalue problems $\rightarrow$ Sensitivity.

$\lambda_i(E)$ $\quad\quad |\lambda_i(A) - \lambda_i(A+E)| \le \dfrac{\|E\|}{|y_i^H x_i|}$

$Ax_i = \lambda_i x_i$, $\quad\quad y_i^H A = \lambda_i y_i^H$

right $\quad\quad$ left

If $A = A^T$, $\quad x_i = y_i$, $\quad\quad \|x_i\| = 1$

$\|y_i\| = 1$

If $A = A^H$, $\quad Q^H A Q = D$

or A is normal $AA^H = A^H A$, $\rightarrow$

$Q^H A Q = D \qquad\qquad Q^H Q = I$

$$\max_i |\lambda_i(A) - \lambda_i(A+E)| \leq \|E\|$$

Ex: 1

$$A = \begin{bmatrix} 4 & 1 & & \\ 1 & -4 & 1 & \\ & 1 & 8 & 1 \\ & & 1 & 20 \end{bmatrix} \qquad \|E\| \leq \frac{1}{4}$$

$$\bar{\lambda}_i = \lambda_i(A+E)$$

$$20 - 1 - \frac{1}{4} \leq \bar{\lambda}_1 \leq 20 + 1 + 1/4$$

$$8 - 2 - \frac{1}{4} \leq \lambda_2 \leq 8 + 2 + 1/4$$

G E

$$4 - 1 - \frac{1}{4} \leq \lambda_{3,4} \leq 4 + 1 + \frac{1}{4}$$

$$-4 - 2 - 1/4 \leq \lambda_{3,4} \leq -4 + 2 + 1/4$$

For general matrices.

$$A \sim \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_p \end{bmatrix} = J, \quad J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_k \end{bmatrix}_{N_k}$$

$$\underline{\underline{S^{-1} A S = J}}$$

$$J_{k+1} = \begin{bmatrix} \lambda_k \end{bmatrix}$$

$$|\lambda_k(A) - \lambda_k(A+E)| \leq C \|E\|^{1/N_k}$$

e.g. $\|E\| \leq 10^{-15}$. $N_k = 3$,

$$|\lambda_i(A) - \lambda_i(A+E)| \leq 10^{-5}$$

<u>Ex:</u>

$$B = \begin{bmatrix} 4 & & & & \\ & 4 & & & \\ & & 5 & -1 & \\ & & & 5 & 1 \\ & & & & 5 \end{bmatrix}$$

$\lambda_i = 4$

$$|\lambda_i(A) - \lambda_i(A+E)| \leq C \|E\|$$

$\lambda_i = 5$

$$|\lambda_i(A) - \lambda_i(A+E)| \leq C \|E\|^{1/3}$$

Solving least squares problems.

Over-determined. $Ax = b$,

$A \in \mathbb{R}^{m \times n}$, $m \geq n$, $\operatorname{rank}(A) = n$.

Ex: Curve fitting

sample

t	t_0	t_1	$\cdots$	t_m
y	y_0	y_1	$\cdots$	y_m

$y(t)$

$$y(t) = a_0$$

or

$$y(t) = a_0 + a_1 t$$

$$y(t) = a_0 \cos t + a_1 \sin t$$

$$\cdots$$

$$y(t) = a_0 + a_1 t + a_2 t^2$$

$\{a_i\}_{i=0}^{n}$ are coefficients to be determined

$$y_n(t) = \sum_{i=0}^{n} a_i \underline{\underline{\varphi_i(t)}}$$

$\{\varphi_i(t)\}$ basis

a_i: coef.

$$y_n(t) = \sum_{i=0}^{n} a_i t^i, \quad \text{polynomial fitting}$$

$$y_n(t_k) = \sum_{i=0}^{n} a_i t_k^i = y_k$$

$n=0$

$$\begin{cases} a_0 = y_0 \\ a_0 = y_1 \\ \vdots \\ a_0 = y_m \end{cases} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [a_0] = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$n=1$

$$A \quad x = b, \quad A \in R^{(m+1)\times 1}$$

$$\begin{cases} a_0 + a_1 t_0 = y_0 \\ a_0 + a_1 t_1 = y_1 \\ \quad \cdots \\ a_0 + a_1 t_m = y_m \end{cases} \qquad \begin{bmatrix} 1 & t_0 \\ 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$A \in R^{(m+1)\times 2} \qquad b \in R^{m+1}$$

In General, $A \in R^{m\times n}$, $m > n$, $\text{rank}(A) = n$

there is no classical soln

i.e. There is no such an x^* that

$$Ax^* - b = 0$$

'The best soln'

$$\|Ax^* - b\| = \min_{x \in R^n} \|Ax - b\|$$

$r(x)$

$1, \checkmark 2, \infty,$

$$\Longleftrightarrow \quad \|Ax^* - b\|_2^2 = \min_{x\in R^n} \|Ax - b\|_2^2$$

$$= \min_{x \in R^n} (Ax - b)^T (Ax - b)$$

$$(x^T A^T - b^T)(Ax - b)$$

$$= \min_{x \in R^n} \{ \underbrace{x^T A^T A x}_{\text{Quadratic form of } A^TA} - x^T A^T b - b^T A x + b^T b \}$$

$$= F(x_1, x_2, \ldots, x_n)$$

It's differentiable.

$$\min_{x\in R^n} \|Ax - b\|_\infty = \min_{x\in R^n} \max_{1\le i\le n} |\underline{(Ax-b)}_i|$$

$$\|Ax-b\|_2^2 = \underline{x^T A^T A x} - b^T A x - x^T A^T b + b^T b$$

$v = \quad v$

$f(x_1, x_2, \dots, x_n)$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

$$\nabla F = \nabla x^T y = \nabla \sum x_i y_i = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\underline{b^T Ax} = (Ax)^T b = x^T A^T b$$

$$F(x) = x^T B x \qquad \nabla F = 2Bx$$

$$= x^T \begin{bmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ a_{21}x_1 + \dots + a_{2n}x_n \\ \\ a_{n1}x_1 + \dots + a_{nn}x_n \end{bmatrix}$$

$$= x_1 a_{11} x_1 + \dots + x_1 a_{1n} x_n$$
$$x_2 a_{21} x_1 + \dots + x_2 a_{2n} x_n$$

$$= \sum a_{ij} x_i x_j \qquad A = A^T$$

$$\nabla F = \begin{bmatrix} 2a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 \\ \\ \end{bmatrix}$$

$$\nabla F = \nabla x^T A^T A x = 2A^T A x$$

$$\nabla \|Ax-b\|_2^2 = 2A^T A x - 2A^T b = 0$$

$$\underline{A^T A} x = A^T b \qquad \text{normal Eqn of } Ax=b$$

$R^{n\times n} \qquad R^{n\times 1}$

$\cdots \quad Ax=b, \quad A\in\mathbb{R}^{m\times n}, \quad m>n,$

$\text{rank}(A)=n,$ full column rank.

Wellposedness $\quad \min_{x\in\mathbb{R}^n} \|Ax-b\|_2$

$F(x_1, x_2, \dots, x_n) \quad \Longleftrightarrow \min \|Ax-b\|_2^2$

$= x^T A^T A x - 2 b^T A^T x + b^T b$

Existence: (A) $\quad \geq 0$ Finite dimension + continuity

(B)

$$A^TAx = A^T b \quad \text{RHS}$$

Normal Eqn

$\forall x\in\mathbb{R}^n, \quad x\neq 0, \quad \underbrace{A^TAx}_{n\times m \ m\times n} \neq 0 \quad A^TA\in\mathbb{R}$

$x^TA^TAx = (Ax)^TAx = \|Ax\|_2^2$

$A(:,i) =$ i-th column $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$Ax = \sum_{i=1}^{n} x_i A(:,i) \neq 0$ if $\vec{x}\neq 0$

Since A has full column rank.

A^TA is invertible.

Soln. exist and it's unique

$$x = (A^TA)^{-1}A^Tb.$$

Exs

t	$t_0 - - -$
$y(t)$	

$$y(t) = \sum_{i=0}^{n} a_i t^i$$

$n=0$ $\quad \underline{y(t) = a_0}$ $\quad A = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $\quad b = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} a_0 = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_m \end{bmatrix}$$

$$(m+1)\, a_0 = \sum_{j=0}^{m} y_j$$

$$a_0 = \frac{1}{m+1} \sum_{j=0}^{m} y_j, \quad \text{average}$$

$n=1$ $\quad y(t) \approx a_0 + a_1 t$

$$A = \begin{bmatrix} 1 & t_0 \\ 1 & t_1 \\ \vdots & \\ 1 & t_m \end{bmatrix}, \quad b = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_0 \\ 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} \qquad \begin{bmatrix} y_0 \\ \vdots \\ y_m \end{bmatrix}$$

$$\begin{bmatrix} m+1 & \sum_{i=0}^{m} t_i \\ \sum_{i=0}^{m} t_i & \sum_{i=0}^{m} t_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{m} y_i \\ \sum_{i=0}^{m} t_i y_i \end{bmatrix}$$

$$a_0 = \frac{\sum y_i \sum t_i^2 - \sum t_i \sum y_i}{(m+1)\sum t_i^2 - \left(\sum_{i=} t_i\right)^2}$$

$$a_1 = \frac{(m+1)\sum t_i y_i - \sum t_i \sum y_i}{(\qquad)}$$

General n $\qquad A^T$

$\sum_{i=0}^{m} t_i^0$

$$A^TA = \begin{bmatrix} m+1 & \sum t_i & \sum t_i^2 \\ \sum t_i & \sum t_i^2 & \sum t_i^3 \\ \sum t_i^2 & \sum t_i^3 & \sum t_i^4 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} \sum_{i=0}^{m} y_i \\ \sum t_i y_i \\ \sum t_i^2 y_i \end{bmatrix}$$

Extension: $y(t) = a_1 \cos t + b_1 \sin t$

$$A = \begin{bmatrix} \cos t_0 & \sin t_0 \\ \vdots & \vdots \\ \cos t_m & \sin t_m \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ \vdots \\ y_m \end{bmatrix}$$

$$A^T A = \begin{bmatrix} \sum_{i=0}^{m} \cos^2 t_i & \sum \cos t_i \sin t_i \\ x & \sum \sin^2 t_i \end{bmatrix}$$

$$A^T y = \begin{bmatrix} \sum_{i=0}^{m} (\cos t_i) y_i \\ \sum_{i=0}^{m} (\sin t_i) y_i \end{bmatrix}$$

Ex 4 $y(t) \simeq a_1 e^{a_2 t}$

$$\log y(t) = \log a_1 + a_2 t$$

$$\widehat{y(t)} = \widehat{a_1} + a_2 t$$

becomes a linear least squares problem.

$$\sqrt{\sum \left(b_i - \sum a_{ij} t_j\right)^2}$$

QR method for least squares problem. Why:

We only use the normal Eqn approach if n is small to modest, $\mathrm{Cond}(A)$ is small,

$$\mathrm{Cond}_2(A^TA) = (\mathrm{cond}(A))^2$$

Eigenvalues of A^TA and AA^T

$$AA^Tx = \lambda x \quad x \neq 0$$

$\lambda(AA^T)$

$$A^TA(A^Tx) = \lambda(A^Tx)$$

$\lambda \neq 0$

$\Rightarrow A^Tx \neq 0$.

Solve $Ax = b$

$$A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}_{m \times n} \qquad R_1 \in \mathbb{R}^{n \times n}, \quad \det(R_1) \neq 0.$$

$R_2 = 0$

$$\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \ddots & \vdots \\ & & & r_{nn} \\ & & R_2 & \end{bmatrix}$$

The least squares solution to this problem

$b_1 \in \mathbb{R}^{n \times 1}, \quad b_2 \in \mathbb{R}^{m-n, 1} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\begin{bmatrix} R_1 \\ 0 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad x = R_1^{-1} b_1$$

$\|x\|_2^2 = \sum x_i^2$

$$\|b - Ax\|_2^2 = \left\| \begin{pmatrix} R_1 x - b_1 \\ -b_2 \end{pmatrix} \right\|_2^2 = \underbrace{\|R_1 x - b_1\|_2^2}_{=0} + \|b_2\|_2^2$$

$x = R_1^{-1} b_1$

min $\|$ $\| \cancel{\#} b_2 \|_2^2$

QR method for least squares problems.

$Ax=b$, $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A)=n$

$\min_{x\in\mathbb{R}^n} \|Ax-b\|_2$

Normal Eqn.

$\underline{A^TA}x = A^Tb$

S.P.D.

$\operatorname{cond}_2(A^TA) = (\operatorname{cond}_2(A))^2$

(ii) QR method

$A = QR$

Need n House

$$P\begin{bmatrix} \text{—} \\ \text{—} \\ \text{—} \\ \text{—} \end{bmatrix} \to \begin{bmatrix} * & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$$

$Ax=b$

$QRx=b$, $Q^HQ=I$

$Rx = Q^Hb$

$$\begin{bmatrix} \times & \times & \times \\ & \times & \times \\ 0 & & \times \\ & & \times \\ & & \times \\ & & \times \\ & & \vdots \end{bmatrix} \quad \begin{bmatrix} \times & \times \\ \times & \times \\ \times & \times \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ 0 \end{bmatrix} x = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix}$$

$$\begin{bmatrix} \times & \times \\ 0 & \times \\ \hline \vdots & \times \\ 0 & \vdots \end{bmatrix}$$

$x_{ls} = R_1^{-1}\tilde{b}_1$

$\begin{bmatrix} A & \vdots & b \end{bmatrix}$ $P_n \cdots P_2 P_1 [A \mid b]$

$$P_1 A = \left[\begin{array}{c|c} R_1 & \hat{b}_1 \\ 0 & \hat{b}_2 \end{array}\right]$$

$R_1 x = \hat{b}_1$, backward subst.

<u>Ex</u> Use the QR to solve $Ax = b$

t_i	1	2	3	4
y_i	0	1	−1	6

$y(t) = a_0 + a_1 t$ Find a_0, a_1

$$\begin{array}{c} A \;\vdots\; b \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \\ 1 & 4 & 6 \end{array}\right] \end{array} \qquad P_1 [A \mid b] = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$w = \frac{x-y}{\|x-y\|_2}$

$$P_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad w = \frac{x-y}{\|x-y\|_2} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} / \sqrt{12}$$

$$P_1 = I - 2\begin{bmatrix}3\\1\\1\\1\end{bmatrix}\begin{bmatrix}3 & 1 & 1 & 1\end{bmatrix}/12$$

$$P_1\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \left(I - \frac{1}{6}\begin{bmatrix}3\\1\\1\\1\end{bmatrix}\underline{\begin{bmatrix}3 & 1 & 1 & 1\end{bmatrix}}\right)\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

$$= \begin{bmatrix}-5\\0\\1\\2\end{bmatrix}$$

$$P_1 b = P_1\begin{bmatrix}0\\1\\-1\\6\end{bmatrix} \cdots\cdots = \begin{bmatrix}-3\\0\\-2\\5\end{bmatrix}$$

$$P_1[A \mid b] = \begin{bmatrix}-2 & -5 & \vert & -3\\0 & 0 & \vert & 0\\0 & 1 & \vert & -2\\0 & 2 & \vert & 5\end{bmatrix}$$

$$\begin{matrix}1 & 0 & 0 & 0\\0 & & & \\0 & & P_2 & \\0 & & & \end{matrix}$$

$$W_2 = \begin{bmatrix}-\sqrt{5}\\1\\2\end{bmatrix}/\sqrt{10}$$

$$P_2\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}+\sqrt{5}\\0\\0\end{bmatrix}$$

$$P_2P_1[A \mid b] = \begin{bmatrix}-2 & -5 & \vert & -3\\0 & \sqrt{5} & \vert & -18/\sqrt{5}\\0 & 0 & \vert & -8/5\\0 & 0 & \vert & 9/5\end{bmatrix}$$

$$\min\|Ax-b\| = \sqrt{\left(\frac{8}{5}\right)^2+\left(\frac{9}{5}\right)^2}$$

x_{LS}

$$x_2 = -18/5,$$

$$x_1 = \left(-3 + 5\cdot\left(-\frac{18}{5}\right)\right)/-2,$$

SVD approach. For any $Ax=b$

e.g. $2x_1 + x_2 - 3x_3 = 7$. $A \in \mathbb{R}^{m\times n}$

$\mathrm{rank}(A) = p$

SVD Thm: $p \le \min\{m,n\}$

Given any $A \in \mathbb{C}^{m\times n}$, there are two orthogonal $U \in \mathbb{C}^{m\times m}$, $U^H U = I = U U^H$

$V \in \mathbb{C}^{n\times n}$, $V^H V = V V^H = I$

$$A = U \Sigma V^H \qquad \underline{U} A V^H$$

$$= \underline{U} \begin{bmatrix} \sigma_1 & & 0 & \\ & \sigma_2 & & \\ 0 & & \ddots \sigma_p & 0 \\ & 0 & & 0 \end{bmatrix}^{m\times n} \underline{V}^H$$

$\sigma_1 \ge \sigma_2 \ge \cdots \sigma_p > 0$ are singular values of A. $\|A\|_2 = \sigma_1$

Pseudo-inverse of A

$$A^+ = V \begin{bmatrix} \frac{1}{\sigma_1} & & 0 & \\ & \frac{1}{\sigma_2} & & \\ 0 & & \ddots \ {}^1\!/\!_{\sigma_p} & 0 \\ & 0 & & 0 \end{bmatrix}^{n\times m} U^H \qquad \begin{aligned} A &= BC \\ A^{-1} &= C^{-1}B^{-1} \end{aligned}$$

$\|A^+\|_2 = \frac{1}{\sigma_p}$, $\mathrm{cond}_2(A) = \dfrac{\max_{1\le i\le p}\{\sigma_i(A)\}}{\min_{1\le i\le p}\{\sigma_i(A)\}}$

σ_i^2 are eigenvalues of $A^H A$

$A A^H$

Proof: $A = \underline{U \Sigma V^H}$

$$A^H A = \underbrace{V \Sigma^H U^H}_{A^H} U \Sigma V^H$$

A similarity transform

$$= V \Sigma^2 V^H = V \begin{bmatrix} \sigma_1^2 & & 0 & \\ 0 & \ddots & \sigma_p^2 & 0 \\ 0 & & 0 & \end{bmatrix} V^H$$