Householder matrix

$$P = I - 2ww^T, \qquad \|w\|_2 = 1 \checkmark$$

$$= I - \frac{2ww^T}{\|w\|_2^2}$$

Thm: $P = P^T$, $P^T P = I$, $P^{-1} = P^T$

P is an orthogonal matrix

$$P^T P = (I - 2ww^T)(I - 2ww^T)$$

$$= I - 2ww^T - 2ww^T + 4w\overbrace{w^Tw}^{=1}w^T$$

$$= I \checkmark$$

Given $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$

Can we find an orthogonal matrix P such that

$$Px = y, \quad P^T = P^{-1}.$$

True if $\|Px\|_2 = \|y\|_2$

$$\|x\|_2 = \|y\|_2.$$

$$P_1 A$$

$$P_1 \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} = \begin{bmatrix} * & & \\ 0 & & \\ 0 & & \\ \vdots & & \\ 0 & & \end{bmatrix}$$

<u>Thm:</u> If $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $\|x\|_2 = \|y\|_2$, then we can find a Householder matrix P (or W) $P^T P = I$ such that

$$Px = y.$$

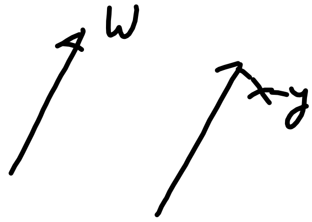
$$W = \pm \frac{x-y}{\|x-y\|_2}$$

$$P \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\alpha = \pm \| \ \|_2 = \pm \sqrt{\sum_{i=1}^{n} a_{i1}^2}$$

Proof: $Px = y$, determine $w = \pm\frac{x-y}{\|x-y\|^2}$

$(I - 2ww^T)x = y$



$x - 2w\underline{w^Tx} = y$

vector

vector

$\underline{x-y} = \underline{(2w^Tx)}\,\underline{w}$

$w \parallel x-y$

$w = \alpha(x-y) = \pm\frac{x-y}{\|x-y\|_2}$

Verify, $w = \frac{x-y}{\|x-y\|_2}$

$(I - 2ww^T)x \overset{?}{=} y$

$= x - 2ww^Tx$

$= x - 2\frac{(x-y)(x-y)^T}{(x-y)^T(x-y)}x \overset{?}{=} y$

$= y + (x-y) - \frac{2\,(}{\qquad)}$

$= y + (x-y)\left(1 - \frac{2x^Tx - 2x^Ty}{(x-y)^T(x-y)}\right) = y$

$(x-y)^T(x-y) = \underline{x^Tx} - x^Ty - y^Tx \underbrace{+y^Ty}_{}$ $\ {}^{x^Tx}$

$= 2x^Tx - 2x^Ty$

Ex: Find P such that $P^T = P^{-1}$

$$P \overset{x}{\begin{bmatrix}3\\0\\4\end{bmatrix}} = \overset{y}{\begin{bmatrix}\alpha\\0\\0\end{bmatrix}}$$

step 1. Find α

$\alpha = \pm 5$

$\sqrt{3^2+4^2} = \sqrt{\alpha^2}$

Step 2 Find ω

$$w = \frac{x-y}{\|x-y\|_2} = \frac{\begin{bmatrix}3-\alpha\\0\\4\end{bmatrix}}{\|x-y\|_2}$$

$\alpha = -5$ to avoid cancellation

$$y = \begin{bmatrix}\alpha\\0\\0\end{bmatrix} = \begin{bmatrix}-5\\0\\0\end{bmatrix}$$

$$W = \begin{bmatrix}3+5\\0\\4\end{bmatrix} / \sqrt{80} = \begin{bmatrix}2\\0\\1\end{bmatrix} / \sqrt{5}$$

$$P = I - 2ww^T = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{5}\begin{bmatrix}2\\0\\1\end{bmatrix}\begin{bmatrix}2&0&1\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{3}{5}&0&-\frac{4}{5}\\0&1&0\\-\frac{4}{5}&0&\frac{3}{5}\end{bmatrix}$$

$I - 2w\underline{w}^T$

$a_1^T a_3$

$P^T P = I$

$\frac{12}{25} \quad -\frac{12}{25}$

QR decomposition of a matrix A

$A \in \mathbb{R}^{n \times n}$, $\det(A) \neq 0$, (not essential condition)

We can find an orthogonal matrix Q

$Q^T Q = I$ such that $\begin{cases} Q^T A = R \\ A = QR \end{cases}$

R is an upper triangular matrix

$$\|A\|_2 = \|R\|_2$$

Proof. Let
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

$$P_1 A = \begin{bmatrix} \tilde{a}_{11} & \times & \times \\ 0 & & \\ 0 & & \\ \vdots & A_2 & \\ 0 & & \end{bmatrix}$$

$$\tilde{a}_{11} = \pm \sqrt{\sum_{i=1}^{n} a_{i1}^2}$$

$$w_1 = \frac{\begin{bmatrix} a_{11} - \tilde{a}_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}}{\| \quad \|_2}$$

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_2 \end{bmatrix} \qquad \underline{P_2} P_1 A = \begin{bmatrix} \tilde{a}_{11} & - & - & - \\ 0 & \tilde{a}_{22} & - & - \\ \vdots & 0 & & \\ 0 & \vdots & A_3 & \\ & 0 & & \end{bmatrix}$$

$$P_{n-1} P_{n-2} \cdots P_1 A = R \qquad \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$$

$$A = (P_{n-1} \cdots P_1)^{-1} R$$
$$= P_1 P_2 \cdots P_{n-1} R = QR$$

Discussions: Can we use QR decomposition to solve $Ax=b$?

Yes. Worth it?

pros. More accurate

con's double's cost of LU.

Destroy the structures,

$$Ax=b$$
$$QRx=b$$
$$Rx=Q^Tb$$

backward substi.

$$P_1 A = \begin{bmatrix} \hat{a}_{11} & - & - \\ 0 & & \\ \vdots & A_2 & \\ 0 & & \end{bmatrix} \quad P_1 = \begin{bmatrix} \hat{a}_{11} \\ \hat{a}_{21} \\ \vdots \\ \hat{a}_{n1} \end{bmatrix}$$

Can we use QR to perform a similarity transform so that A becomes diagonal matrix. $Q^T A Q \overset{?}{=} D$

Thm: For any matrix $A \in \mathbb{C}^{n \times n}$. we can find the QR decomposition

$A = QR$. R is an upper triangular matrix. $Q^H Q = Q Q^H = I$.

$A = LU \longrightarrow PA = LU$

$Ax = b$, $QRx = b$

$Rx = Q^H b$

$\|R\|_2 = \| \; \| = \|A\|_2$

QR needs more computation (doubled)

destroy the structures of A.

$Q^H Q = I,$

$\max_{1 \le i,j \le n} |q_{ij}| \le 1$

$|a+bi| = \sqrt{a^2+b^2}$

$\sum_{j=1}^{n} q_{ij} \bar{q}_{ij} = \sum_{i=1}^{n} |q_{ij}|^2 = 1$

$|q_{i_0,j_0}|^2 \le \sum_{i=1}^{n} |q_{ij}|^2 = 1$

$L L^T$

$\sum l_{ij}^2 = a_{ii}$

Ex: $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 25 \\ 4 & 5 & 0 \end{bmatrix}$

Find the QR decomposition.

Unique e^{ik}

$$P_1 \overset{x}{\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}} = \overset{y}{\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} \pm 5 \\ 0 \\ 0 \end{bmatrix}$$

$0^2 + 3^2 + 4^2 = \alpha^2$

$25 = \alpha^2 \quad \alpha = \pm 5$

$$W_1 = \frac{x-y}{\|x-y\|_2} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} \Big/ \sqrt{25+9+16}$$

$$P_1 A = [P_1 a_1, P_1 a_2, P_1 a_3]$$

$$= \begin{bmatrix} -5 & -4 & -15 \\ 0 & -3 & 16 \\ 0 & 1 & -12 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{I - 2W_2 W_2^T} \\ 0 & & \end{bmatrix}$$

$$P_1 \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = (I - 2W_1 W_1^T) \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} - \frac{2}{50} \cdot 25 \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$W_1^T \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = [5 \ 3 \ 4] \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

$$= 25$$

$$P_1 \begin{bmatrix}0\\25\\0\end{bmatrix} = \left(I - \frac{2w_1w_1^T}{50}\right)\begin{bmatrix}0\\25\\0\end{bmatrix}$$

$$P_1 a_3 = \begin{bmatrix}0\\25\\0\end{bmatrix} - \frac{1}{25}\underset{w_1}{\begin{bmatrix}5\\3\\4\end{bmatrix}}\ \overset{w_1^T}{\begin{bmatrix}5 & 3 & 4\end{bmatrix}}\begin{bmatrix}0\\25\\0\end{bmatrix}$$

$$25\cdot 3$$

$$= \begin{bmatrix}-15\\16\\-12\end{bmatrix}$$

$$P_2P_1A = P_2\begin{bmatrix}-5&-4&-15\\0&-3&16\\0&1&-12\end{bmatrix}$$

$$+$$

$$\tilde{P}_2\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}\beta\\ \gamma\end{bmatrix} = \begin{bmatrix}\pm\sqrt{10}\\0\end{bmatrix}$$

$$= \begin{bmatrix}-5&-4&-15\\0&\sqrt{10}&x\\0&0&x\end{bmatrix} = R$$

$$w_2 = \frac{x-y}{\|x-y\|_2} = \begin{bmatrix}-3-\sqrt{10}\\1\end{bmatrix} \Big/ \sqrt{(3+\sqrt{10})^2+1}$$

$\underline{w_i^T}(\,)$

QR method for algebraic eigenvalue problems: find all eigenvalues.

Start with $A_0 = A$, $A \in R^{n\times n}$

$A_0 = Q_0 R_0$, $Q_0^H Q_0 = I$, R_0 is an upper tria.

$A_1 = R_0 Q_0$

$A_1 = Q_1 R_1$ $Q_1^H Q_1 = I$. R_1 is upper

$A_2 = R_1 Q_1$

...

QR method

$$\begin{cases} A_k = Q_k R_k & Q_k^H Q_k = I,\ R_k \text{ upper tria} \\ A_{k+1} = R_k Q_k & k = 0, 1, \dots, \text{until converge.} \end{cases}$$

<u>Thm</u>: From the QR method, we have

If $A \in R^{n\times n}$, $|\lambda_{k1}| > |\lambda_{k2}| > \dots > |\lambda_{ks}|$ $1 < s < n$.

(i) $A_{k+1} \sim A_k \sim A$ $|\lambda_{k1}|$ if λ_{k1} is real.

$|\overline{\lambda_{k2}}|$ if λ_{k1} and λ_{k1} is a complex pair.

$\underline{S^{-1} A_{k+1} S = A_k}$

A_{k+1}, A_k, and A all have the same eigenvalues.

(ii) $\lim_{k\to\infty} A_k = R_A = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1s} \\ & R_{22} & & \vdots \\ & & \ddots & \\ 0 & & & R_{ss} \end{bmatrix}$

a block upper triangular matrix

Where R_{kk} is either a 1 by 1 or 2 by 2 matrix $\begin{bmatrix} y & x \\ x & x \end{bmatrix}$

corresponding to either a simple or complex eigenvalue pair.

$\det(\lambda I - A) = \prod \det(\lambda_k I - R_{kk}) = 0$

Schur form of A

QR method $A_0 = A \in \mathbb{R}^{n \times n}$.

$A_0 = Q_0 R_0$

$A_1 = R_0 Q_0$

$$\begin{cases} A_k = Q_k R_k & \rightarrow Q_k^T A_k Q_k \\ A_{k+1} = R_k Q_k & = \underline{Q_k^T Q_k} \, \underline{R_k \, Q_k} \end{cases}$$

$= I \; A_{k+1}$

$A_{k+1} \sim A_k \sim A_{k-1} \cdots = A_0 = A$ $\quad = A_{k+1}$

$$A_k \longrightarrow \begin{bmatrix} R_{11} & R_{12} & & R_{1s} \\ & R_{22} & & \vdots \\ & & \ddots & \\ 0 & & & R_{ss} \end{bmatrix}$$

$$\left| \frac{\lambda_k}{\lambda_{k-1}} \right|$$

$$\begin{bmatrix} \lambda_i & & 0 \\ & \lambda_i & \\ 0 & & \ddots \; \lambda_i \end{bmatrix}$$

$$\begin{bmatrix} * \\ 0 \\ \vdots \end{bmatrix}$$

Shifted QR method $A = A_0$

for $k = 0, 1, \ldots$ until converges

$$\begin{cases} A_k - \sigma_k I = Q_k R_k \\ A_{k+1} = R_k Q_k + \sigma_k I \end{cases}$$

end

Stop criteria:

$$\max_{\substack{3 \le i \le n \\ 1 \le j \le i+2 \le n}} |a_{ij}| < tol$$

QR method is expensive. $\times \div$

each iteration $A_k = Q_k R_k$ $O(\frac{2}{3}n^3)$

$A_{k+1} = R_k Q_k$ $O(\frac{n^3}{2})$

Can we use QR method to reduce A to a diagonal matrix? In finite operations

$A \xrightarrow[\text{transform}]{\text{similarity}} D$ No if A is not diagonalizable, e.g. Jordan Block

1. $A = A^T$, $A = A^H$
2. A has complete eigenvectors

We can use similarity transform to reduce A to an upper Hessenberg matrix.

$$H = \begin{bmatrix} x & x & x & \cdots & x \\ x & x & x & x & x & \cdots \\ 0 & x & \ddots & & \ddots & \\ 0 & 0 & \ddots & \ddots & & \checkmark \\ \vdots & \vdots & & \ddots & \ddots & x \\ 0 & \cdots & & & x & x \end{bmatrix}$$

in $n-2$ transforms.

If $A=A^T$, we can transform A to a tridiagonal matrix.

Step 1.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix} \quad n \geq 2$$

$$P_1 A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \hat{a}_{21} & \\ 0 & \\ \vdots & A_2 \\ 0 & \end{bmatrix} \qquad P_1 = \begin{bmatrix} 1 & 0 \cdots 0 \; 0 \\ & I_{n-1} - \frac{2 w_1 w_1^T}{\|w_1\|^2} \end{bmatrix}$$

$$w_1 = \begin{pmatrix} a_{21} \pm \tilde{a}_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{pmatrix} \qquad |\tilde{a}_{21}| = \pm\sqrt{\sum_{i=2}^{n} |a_{i1}|^2}$$

$$\tilde{a}_{21} = -\operatorname{sgn}(a_{21}) \cdot \sqrt{\sum_{i=2}^{n} |a_{i1}|^2}$$

$$P_1 A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ 0 & \hat{a}_{32} & & \\ \vdots & \vdots & & \\ 0 & \hat{a}_{n2} & & \hat{a}_{nn} \end{bmatrix}$$

$\tilde{P}_1 A$

$$\begin{bmatrix} \hat{a}_{11} & \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix} \tilde{P}_1$$

$$P_1 A P_1 = \begin{bmatrix} a_{11} & & \\ \hat{a}_{21} & & \\ 0 & \times & \\ \vdots & & \\ 0 & & \end{bmatrix}$$

$$\begin{bmatrix} \hat{a}_{11} & \\ \vdots & \\ \times & \end{bmatrix}$$

$P_1 = P_1^T = P_1^{-1}$

Similarity transform.

$$P_{n-2} \cdots P_2 P_1 A P_1 P_2 \cdots P_{n-2}$$

In finite number of operations.

$$PAP^T = H.$$

If A is symmetric $A = A^T$

$$(PAP^T)^T = H^T$$

$$PA^TP^T = PAP^T = H^T \Rightarrow H^T = H$$

$$H = \begin{bmatrix} a_{11} & a_{21} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{32} & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33} & & & \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & 0 & & \ddots & \ddots & a_{n,n-1} \\ & & & & a_{n,n-1} & a_{nn} \end{bmatrix}$$

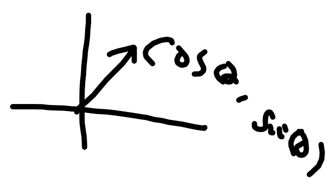
Then we apply the QR method.

QR method for upper Hessenberg matrix.

$$Q_1 \quad H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ \underline{h_{21}} & h_{22} & \cdots & \\ 0 & \ddots & \ddots & \\ \vdots & & h_{n\,n-1} & h_{nn} \end{bmatrix}$$

Given's rotation

$\rightarrow h_{21} \rightarrow 0.$ divide - conquer



Givens' matrix.

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}.$$

$$Q^T Q = I.$$

$q_{21}^2 \le q_{11}^2 + q_{21}^2 = 1$

$q_{12}^2 + q_{22}^2 = 1$

$q_{11} q_{12} + q_{21} q_{22} = 0$

$|q_{ij}| \le 1$ for all i, j

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

<u>Ex.</u>: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$P_1 = I - \frac{2 w_1 w_1^T}{\|w_1\|_2^2}$$

$$QA = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \quad \pm\sqrt{2^2+1} = x^2$$

$$Q = \begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{pmatrix} P_1$$

$$= \begin{bmatrix} \pm\sqrt{5} & * \\ 0 & * \end{bmatrix}$$

<u>Real</u>: $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} P_1$ real.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \pm\sqrt{5} & * \\ 0 & * \end{bmatrix}$$

$$-2\sin\theta + \cos\theta = 0 \qquad \tan\theta = \frac{1}{2}$$

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta} \qquad 1 + \left(\frac{1}{2}\right)^2 = \frac{1}{\cos^2\theta}$$

$$\cos\theta = \pm\frac{2}{\sqrt{5}} \overset{\text{take}}{=} \frac{2}{\sqrt{5}}$$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\frac{1}{\sqrt{5}}$$

Take:

$\cos\theta = \frac{2}{\sqrt{5}}$

$\sin\theta = \frac{1}{\sqrt{5}}$

verify: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & \frac{4}{\sqrt{5}} \\ 0 & \frac{3}{\sqrt{5}} \end{bmatrix}$$

If we want to reduce one entry to zero, then we should use the Givens' rotation.

A. Want to find all eigenvalues, we can use the QR method. It's computationally expensive. Often we prefer

$$A \xrightarrow[\text{transform}]{\text{Householder}} H \quad \text{the upper Hessenberg form.}$$

Need $n-2$ steps.

Then we apply Givens' rotations for the QR method.

QR

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{G_{12}} \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$G_{ij} = \begin{array}{c} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \cos\theta & \sin\theta & \\ & & \sin\theta & \cos\theta & \\ & & & & \ddots \; 1 \end{bmatrix} i \\ j \end{array} \xrightarrow{G_{23}} \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{G_{34}} \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} = R$$

To transfer H to an upper triangular matrix using Given's rotation. requires $O(n^2)$ operations.

- Generalized algebraic eigenvalue problems

$$Ax = \lambda Bx. \rightarrow A'x = \lambda x$$

- Perturbation Theory A $A+E$

$\|E\| \le c\|A\|$ $Ax = \lambda x$ $(A+E)\bar{x} = \bar{\lambda}\bar{x}$

J. H. Wilkinson's backward error analysis.

<u>Thm.</u> If A is diagonalizable, (A has completed eigenvectors), all Jordan Blocks are 1 by 1.

$$A \sim \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 3 \end{bmatrix}$$

We can find S, $\det(S) \neq 0$, such that

$$S^{-1}AS = D$$

$$A + E$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\min|\lambda_i - \bar{\lambda}_i| \le \underline{\overset{=1}{\text{cond}(s)}}\,\|E\|$$

$$S^{-1} = S^T$$

If $\underline{AA^T = A^TA}$, A is called a normal matrix, then $\text{cond}(s) = 1$.

If $A = A^T$, then is a normal matrix!