The Power method in x_p form.

'different scaling'

x_p notation: Given $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Define x_p as the 1-st component such that $|x_p| = \|x\|_\infty$, p is the index. e.g.

$$x = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \qquad x_p = -3, \quad p = 3$$

$$\|x\|_\infty = 3$$

Power method II

$Tol = 10^{-6}$

$err = 10^{3}$

Given x_0,

for $k=1$ until converges

$y_{k+1} = A x_k$ → the same

$x_{k+1} = y_{k+1} / (y_{k+1})_p$ $\quad \|x_{k+1}\|_\infty = 1$

$\mu_{k+1} = (y_{k+1})_p$ $\quad x_{k+1} = \frac{A^{k+1} x_0}{\alpha}$

end

1 Q: How about $\|\cdot\|_1$ norm?

Convergence speed, linear $\left|\frac{\lambda_2}{\lambda_1}\right|$ ✓

quadratic $\left|\frac{\lambda_2}{\lambda_1}\right|^2$.

If $A = A^T$, then the original Power method is quadratically convergent, otherwise it is linearly convergent.

linear convergent

$$x_{k+1} = \alpha_k A^{k+1} x_0$$

$$= \alpha_k \left(\beta_1 v_1 + \beta_2 v_2 \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1}}_{\to 0} + \cdots + \beta_n v_n \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^{k+1}}_{\to 0} \right)$$

$A v_i = \beta_i v_i, \quad i = 1, 2, \ldots, n.$

$A, \quad \lambda_1, \lambda_2, \ldots, \lambda_n$

$A - \sigma I \qquad \lambda_1 - \sigma, \ \lambda_2 - \sigma, \ldots, \lambda_n - \sigma$

$\det(\lambda I - (A - \sigma I)) = 0$

If we want to approximate λ_k, we can find a σ such that

$$|\sigma - \lambda_k| = \min_{1 \le i \le n} |\sigma - \lambda_i(A)|$$

$\lambda_i(A) \quad 9 \quad 8 \quad 7 \quad 5 \quad 2 \quad 0 \quad -1$

$\sigma = 4.6 \quad 4.4 \quad 3.4 \quad 2.4 \quad 0.4 \quad 1.4$

$\sigma - \lambda_i(A)$

$\underline{\underline{0.4}}$

We use the <u>shifted Power</u> method

For $k=1$ until converges

$$(A-\sigma I)\, y_{k+1} = x_k$$

$$x_{k+1} = y_{k+1} / \|y_{k+1}\|_2$$

$$\mu_{k+1} = x_{k+1}^T A\, x_{k+1}$$

end

$x_k \to v_k$

$\mu_k \to \lambda_k$

How do we roughly locate eigenvalues.

$$D = \begin{bmatrix} 5 & & & \\ & -4 & & 0 \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} \qquad \lambda_i(A) = a_{ii}.$$

$$\underline{\underline{D+E}} = \begin{bmatrix} 5 & 0{,}01 & & \varepsilon \\ -0{,}03 & -4 & \cdots & \\ & & & \end{bmatrix}$$

$$|\lambda_i(A) - \lambda_i(A+E)| \le C\|E\|,$$ perturbation of eigenvalues.

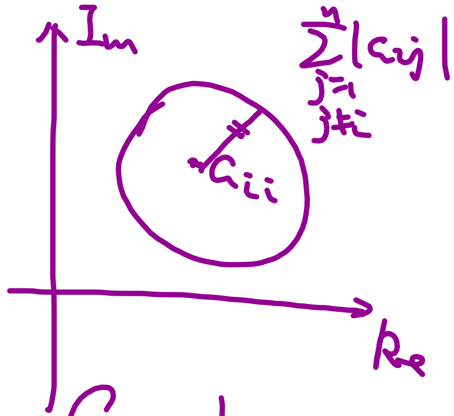
$$Ax = b$$

$$(A+E)\hat{x} = b + \delta b$$

$$\frac{\|x - x_e\|}{\|x_e\|} \le$$

$A \in \mathbb{C}^{n \times n}$,

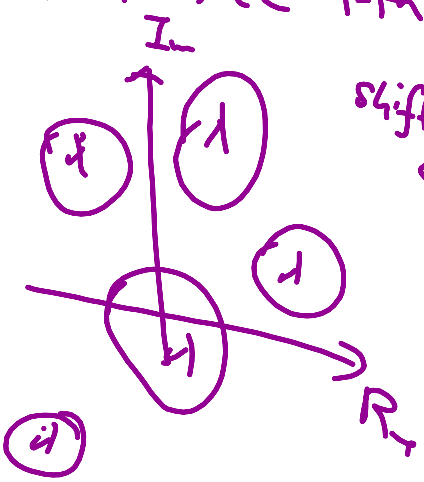
$$|\lambda - a_{ii}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$$



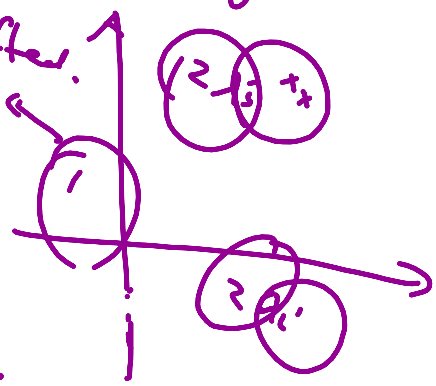
The set is called the ith Gershgorin circle.

Ex

shifed for any



shifted.



Th: Gershgorin.

1. Any eigenvalue has to be in one of Gershgorin circle
2. The union of k Gershgorin circles, which do not intersect with other $n-k$ circles, contains precisely k eigenvalues of A.

Proof: $Ax^* = \lambda^* x^*$, $x_p^* = 1$

$$(Ax^*)_p = \lambda^* x_p^*$$

$$\sum_{j=1}^{n} a_{ij} x_j^* = \lambda^* x_p^*$$

$$\left|(a_{pp} - \lambda^*) x_p^*\right| = \left|\sum_{\substack{j=1 \\ j \neq p}}^{n} a_{ij} x_j^*\right|$$

$$\left|a_{pp} - \lambda^*\right| \leq \sum_{\substack{j=1 \\ j \neq p}}^{n} |a_{ij}|$$

Wellposedness of algebraic eigenvalue problems. $Ax = \lambda x$

existence ✓, uniqueness ✓, sensitivity. $x \neq 0$

$\det(\lambda I - A) = 0$, $\lambda^n \pm (\underbrace{\quad}_{\sum_{i=1}^n a_{ii}})\lambda^{n-1} + \cdots + (\ )\lambda + (\underbrace{\quad}_{\det(A)}) = 0$

trace

Solvability.

If $\det(A) = 0$, $\lambda = 0$ is an eigenvalue. A^{-1} does not exist.

$(A+E)x = \lambda x$ $\qquad \|E\| \leq C \|A\| \varepsilon$

$Ax = \lambda x$, x is called the right eigenvector. $\|x\|=1$

$\|y\|=1$

$y^T A = \lambda y^T$, y is called the left eigenvector. If $A = A^T$, then $x = y$.

$$|\lambda_i(A) - \lambda_i(A+E)| \leq \frac{C\|E\|}{y^T x}$$

$\left|\frac{1}{y^T x}\right|$ is called the condition number of the eigenvalue problem.

$\underline{A = A^T} \in \mathbb{R}^{n \times n}$, then $|\lambda_i(A) - \lambda_i(A+E)| \leq C\|E\|$

Well-conditioned.

eigen-vectors.

$\lambda_i(A)$ are all real. $Q = [x_1, x_2, \dots x_n]$

$Q^T Q = I$ $\quad Q^T A Q = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$ $\quad x_i^T x_j = \delta_i^j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$A = A^T$, then the Power method is quadratically convergent.

$$\mu_k = x_k^T \underline{A x_k} = x_k^T A\left(\beta_k \alpha_1 v_1 + \bar{\beta}_k \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k + \cdots\right.$$

$$= \left(\underline{\beta_k \alpha_1 v_1} + (\cdots)\right)^T \left(\underline{\beta_k \alpha_1} \underline{\underline{\lambda_1}} v_1 + ? \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 \lambda_2\right.$$

$$= \lambda_1 + \bar{\beta}_k \left(\frac{\lambda_2}{\lambda_1}\right)^{2k} v_2 + (\qquad) \to \lambda_1$$

Ex. of Gershgorin Thm.

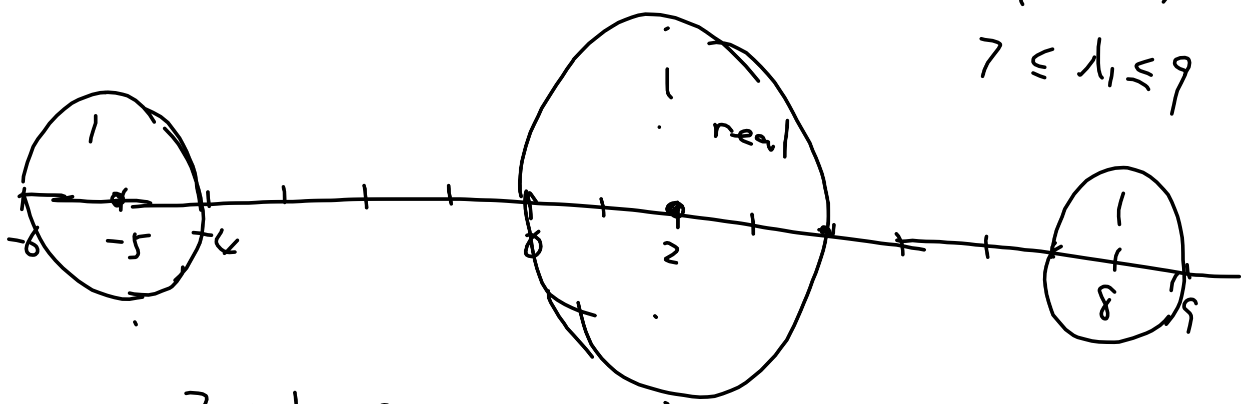
$$A = \begin{bmatrix} -5 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 8 \end{bmatrix}$$

R_1: $|z+5| \leq 1$

R_2 $|z-2| \leq 2$

$|z-8| \leq 1$,

$7 \leq \lambda_1 \leq 9$



$7 \leq \lambda_1 \leq 9$, $0 \leq \lambda_3 \leq 4$, $-6 \leq \lambda_2 \leq -4$

$$(A - \sigma I)\, y_{k+1} = x_{k+1} \qquad \sigma = -5$$

$(A+5I) y_{k+1} = x_{k+1}$. $x_{k+1} = y_{k+1} / \|y_{k+1}\|_2$

$\mu_{k+1} = x_{k+1}^T A x_{k+1}$

$$x_{i-1} - 2x_i + x_{i+1} = f(x_i) \quad - \cos \frac{k\pi}{n+1},$$

$k = 1, 2 \ldots$

Find all eigenvalues, $\rightarrow$ QR method for eigenvalue problems. iterative method

$A \rightarrow \dots \quad D$ A and D should have same eigenvalue

$\underline{S}^{-1} A S = D \quad \det(S) \neq 0.$

Similarity transformation, Not practical for general S, except for orthogonal matrices, $Q^T = Q^{-1}$

$\mathrm{Cond}_2(QA) = \mathrm{cond}_2(A)$

$\|QA\|_2 = \|A\|_2$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\|x\|_2 = \|y\|_2$$

$$QA = \begin{bmatrix} \bar{a}_{11} & \\ 0 & \\ \vdots & \bar{a}_{ij} \\ 0 & \end{bmatrix}$$

Q^T

$$\begin{array}{ccc} x & & y \\ Q\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} & = & \begin{bmatrix} \bar{x}_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array}$$

$$Q^T Q = I.$$

$$\bar{x}_1 = \pm\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Householder matrix in R^n

$$P = I - 2ww^T \qquad I \in R^{n \times n}$$

$$\|w\|_2 = 1$$

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} [w_1 \; w_2 \cdots w_n]$$

$$\begin{bmatrix} w_1^2 & w_1 w_2 & \cdots & w_1 w_n \\ & & & \end{bmatrix}$$