Summarize convergence of J, G-S, SOR(ω)

General matrices:

SCDD J, G-S

Weakly CDD, + irreducible G-S, J.

A: SPD, G-S, SOR(ω), $0<\omega<2$

→ Poisson eqn. (1D, 2D) $D=\sigma I$ $A=D-L-U$

$$\rho(R_J) = \max_{1\le i\le n} \left| 1 + \frac{\lambda_i(A)}{2} \right|$$

$$\det(\lambda I - D^{-1}(L+U)) = 0 \longrightarrow A$$

. . .

$D=\sigma I$

$$\det(\lambda D - (L+U)) = 0$$

$$A = \begin{bmatrix} 2 & & \\ & \ddots & \\ & & 2 \end{bmatrix}$$

. . .

$$\det(\underline{2(\underline{\underline{\lambda}}-1)} I - A) = 0$$

$$\det(\lambda' I - A) = 0$$

$$\lambda_i'(A) = 2(\lambda_i(R) - 1)$$

$$\lambda_i(R) = 1 + \frac{\lambda_i(A)}{2}$$

Lemma: Eigenvalues of

$$A = \begin{bmatrix} \alpha & \beta & & & 0 \\ \beta & \alpha & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \beta \\ 0 & & & \beta & \alpha \end{bmatrix}_{n\times n}.$$

are $\lambda_k = \alpha + 2\beta \cos \frac{\pi k}{n+1}$, $k = 1, 2, \dots n$.

$x_{k,j} = \sin \frac{k\pi j}{n+1}$, $j = 1, 2, \dots, n$.

Check $A \breve{X}_k = \lambda_k \breve{X}_k$.

$$-(p u')' + q u = \lambda u, \quad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0.$$

$$A = \begin{bmatrix} -2 & & & \\ 1 & \ddots & 1 & 0 \\ 0 & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}$$

$h =$

$$\lambda_k = -2 + 2\cos\frac{\pi k}{n+1} \quad \pi/2$$

$Ax = \lambda x$

$$\mathrm{cond}_2(A) = \kappa(A) = \frac{\max|\lambda_i(A)|}{\min|\lambda_i(A)|}$$

$A^T A = A^2 x = \lambda^2 x$

≈ 4

$$\lambda_k = -\cancel{4}\left(1 - \cos\frac{\pi k}{n+1}\right)$$

$$= -\cancel{4}\sin^2\frac{\pi k}{2(n+1)}$$

$$\mathrm{cond}_2(A) = \frac{4}{4}\frac{(n+1)^2}{(\pi h)^2}$$

$$\approx -\cancel{4}\left(\frac{\pi k}{2(n+1)}\right)^2$$

$\vdots$

$$\sim (n+1)^2$$

$$\rho(R_J) = \max_{1 \le k \le n} \left| 1 + \frac{\lambda_k(A)}{2} \right| \qquad 1 + \frac{\lambda_i(A)}{2}$$

$$= \max_{1 \le k \le n} \left| 1 - \frac{2\left(1 - \cos\frac{\pi k}{n+1}\right)}{2} \right|$$

$$= \max_{1 \le k \le n} \left| \cos\frac{\pi k}{n+1} \right| < 1 \qquad k = 1, 2, \ldots$$

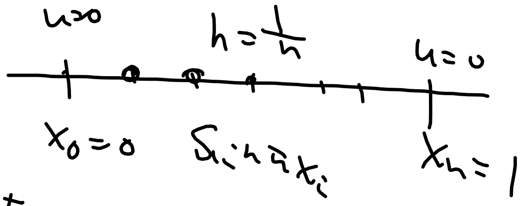
$$\cos\frac{\pi}{n+1} = 1 - \left(\frac{\pi}{n+1}\right)^2$$

The Jacobi iterative method needs $O(n^2)$ for convergence. It's slow, but simple!

$-u'' = \lambda u$ $\quad$ $u=0$, trivial soln.

$u(0) = u(1) = 0$ $\quad$ S-L eigenvalue prob.

✓ $\lambda_n = (n\pi)^2$, $\quad u(x) = \sin n\pi x$, $\quad n = 1, 2, \dots$



$\sin n\pi x_i$

$\sin \frac{\pi x_i}{h}$

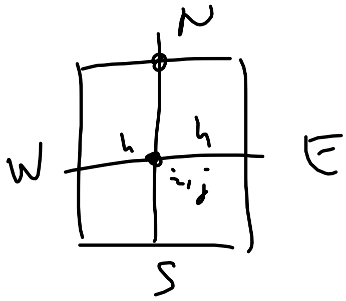
$$\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix}_{n-1, n-1}$$

$\lambda_k = -2 + 2\cos \frac{\pi k}{n}$, $\quad k = 1, 2, \dots, n-1$

$x_{k,j} = \sin \frac{\pi k j}{n}$, $\quad j = 1, \dots, n-1$.

2D: $u_{xx}+u_{yy} = f(x,y)$

$= \lambda u$



$$\frac{U_{i-1,j}+U_{i+1,j}+U_{i,j-1}+U_{i,j+1}-4U_{ij}}{h^2} = f_{ij}$$

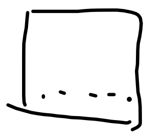
$i,j=1,2,\dots,n-1.$

$$U_{ij} = \frac{1}{4}\Big(\qquad \Big)$$

$$\lambda_{ij} = -\left(4-2\left(\cos\frac{\pi i}{n}+\cos\frac{\pi j}{n}\right)\right)$$

$i,j=1,\dots,n-1.$

$$\rho(R_J) = \max_{1\le i,j\le n-1}\left|1+\frac{\lambda_{ij}(A)}{4}\right|$$



$$= \max_{1\le i,j\le n-1}\left|1-\frac{1}{4}\left(4-2\left(\cos\frac{\pi i}{n}+\cos\frac{\pi j}{n}\right)\right)\right|$$

$$= \max_{1\le i,j\le n-1}\frac{1}{2}\left|\cos\frac{\pi i}{n}+\cos\frac{\pi j}{n}\right|$$

$$= \cos\frac{\pi}{n} \sim 1-O\left(\frac{1}{n^2}\right)$$

$$\rho(R_J) = \max\left|1+\frac{\lambda_i(A)}{2}\right| \quad \text{1D}$$

$$\left|1+\frac{\lambda_i(A)}{4}\right| \quad \text{2D}$$

The Best ω: SOR(ω), $0 < \omega < 2$.

This is a necessary condition for SOR(ω) to converge. $\omega = -0.1$, 2.1 $\rightarrow$ diverge!

For general matrix A, we don't know the best ω. For Poisson Eqn. We know the best ω

$$0 < \omega_{opt} = \frac{2}{1+\sqrt{1-\rho^2(R_J)}} < 2$$

Poisson Eqn. 1D $\dfrac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f_i$

$i = 1, 2, \dots n-1.$ $\begin{cases} U_0 = \alpha \\ u_n = \beta \end{cases}$

2D $\dfrac{U_{i-1,j} + \dots - 4U_{ij}}{h^2} = f_{ij}$ $h \to \dfrac{1}{n+1}$

$D = -2I,$ $D = -4I$ $-A$ is SPD

$$\rho(R_J) = \max \left| 1 + \frac{\lambda_i(A)}{2} \atop \frac{\lambda_i(A)}{4} \right| < 1$$

The No. of iterations for $\begin{matrix} J. \\ G\text{-}S \end{matrix}$ is $O(N^2)$

Thm For $\begin{matrix} 1D \\ 2D \end{matrix}$ Poisson eqns $SOR(\omega_{opt})$ is $O(N)$

(A): $\left(\lambda(SOR(\omega)) + \omega - 1\right)^2 = \lambda(SOR(\omega)) \cdot \omega^2 \lambda_J^2$

(B): $\omega_{opt} = \dfrac{2}{1 + \sqrt{1 - \rho^2(R_J)}}$ $D = -4I$

Key: $R_{SOR} = (D - \omega L)^{-1} \left((1-\omega) D + \omega U \right)$

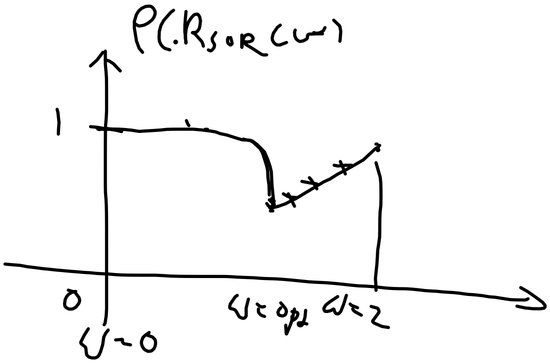
$\boxed{\text{FFT}}$
$\cancel{O(n^2} \log n)$
$\sim O(n^2)$
Multigrid.

$\rho(R_{SOR})$ We would rather choose ω larger than smaller.

$0 < \omega < 2$ for any A.

$$\rho(R_{SOR}) = \begin{cases} 1 - \omega + \frac{1}{2}\omega^2 \rho(R_J)^2 + \omega \rho(R_J) \sqrt{1 - \omega + \frac{\omega^2}{4}\rho(R_J)^2} & 0 < \omega < \omega_{opt} \\ \omega - 1 & \end{cases}$$

$\rho(R_{SOR}) < 1$ if $0 < \omega < 2$.



$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \cos^2 \frac{\pi}{n}}} = \frac{2}{1 + \frac{\pi}{n+1}}$$

Other extensions: $JOR(\omega)$

$$x^{k+1} = (1-\omega)x^k + \omega \bar{x}_J^{k+1}$$

$i = 1, 2, \ldots n \rightarrow SOR(\omega)$

Then $i = n, n-1, \ldots 1$. alternatively.

SSOR symmetric $SOR(\omega)$.

Algebraic eigenvalue problems.

Given $A \in R^{n\times n}$, if there is $\lambda \in C$,

and $x \in R^n$, $x \neq 0$, such that

$Ax = \lambda x$. (λ, x) is eigen-pair of A.

λ can be found theoretically by solving

$$\det(\lambda I - A) = (\underset{\neq 0}{\quad})\lambda^n + (\overset{\text{Trace of } A.}{\quad})\lambda^{n-1} + \dots + (\,)\lambda + (\underset{\det(A)}{\quad}) = 0$$

The characteristic polynomial of A.

There are

$P_n(x) = 0$, roots

$\lambda_1, \lambda_2, \dots, \lambda_n$ $f(x) = 0$. zero

can be multiples eigenvalues of A.

$\lambda_1 = 1 - 2i$

$\lambda_2 = 1 + 2i$ Complex roots are in pair

If λ is an eigenvalue of A, what are eigenvalues of $A^2, \dots, A^k, \dots A^{-k}$ $\frac{1}{\lambda^k}$

$Ax = \lambda x$, $A^2 x = \lambda A x = \lambda \lambda x = \lambda^2 x$

$\mathrm{Cond}(A^T A) = \mathrm{Cond}(A^2) = \|A^2\|_2 \, \|(A^2)^{-1}\|_2$

$= \lambda_{max} |\lambda_i^2(A)| / \min |\lambda_i^2(A)|$

$A^{-1}, (A^{-1})^2,$

$Ax = \lambda x$ $x = \lambda A^{-1} x$

$\frac{1}{\lambda} x = A^{-1} x$

If $\lambda = 0$, A is singular. Can find nonzero $x \neq 0$, $Ax = 0$.

$$\mathrm{Cond}_2(A) = K(A) = \frac{\max |\lambda_i(A)|}{\min |\lambda_i(A)|}$$

$$A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \underline{\lambda_1 = \alpha}$$

$$A = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix}$$

$$\det \begin{pmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{pmatrix} = 0$$

$$\lambda_i(A) = \{\lambda_i(D_1)\} \cup \{\lambda_i(D_2)\}$$

$A \in R^{n \times n}$. assume A has a complete eigen system, or A is diagonalizable

$v_1, v_2, \dots, v_n$

$A v_i = \lambda_i v_i, \quad i = 1, 2 \dots n \quad \|v_i\| = 1$

$$Q^{-1} A Q = D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$Q^T A Q$

$\underbrace{|\lambda_1|}_{=\rho(A)} > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$

essential

observation: $x^0 \neq 0$, $\|x^0\| = 1$.

$x^1 = Ax^0$, $x^2 = Ax^1$, $\ldots$ $x^{k+1} = Ax^k$
$= \cdots = A^{k+1} x^0$

$$x^0 = \sum_{j=1}^{n} \alpha_j v_j$$

$$x^1 = Ax^0 = \sum_{j=1}^{n} \alpha_j A v_j = \sum_{j=1}^{n} \alpha_j \lambda_j v_j$$

$$x^2 = \sum_{j=1}^{n} \alpha_j \lambda_j^2 v_j$$

$\ldots$

$$x^{k+1} = \sum_{j=1}^{n} \alpha_j \lambda_j^{k+1} v_j$$

$\left|\frac{\lambda_i}{\lambda_1}\right| < 1$, $i = 2 \cdots n$.

$$= \lambda_1^{k+1} \Big(\alpha_1 v_1 + \alpha_2 \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1}}_{\to 0} v_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^{k+1} v_n \Big)$$

$x^{k+1} \approx \lambda_1^{k+1} \alpha_1 v_1 \longrightarrow \infty$ if $|\lambda_1| \geq 1$

$Ax^{k+1} = \lambda_1^{k+2} \alpha_1 \longrightarrow 0$ if $|\lambda_1| < 1$.

Soln: Scaling.

Start from x^0, $\|x^0\|_2 = 1$. $\begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{2}$

$$\begin{cases} y_1 = A x_0 \\ x_1 = y_1 / \|y_1\|_2 \\ \mu_1 = x_1^T A x_1 \end{cases}$$

\mu

$$\begin{cases} y_2 = A x_1 \\ x_2 = y_2 / \|y_2\|_2 \\ \mu_2 = x_2^T A x_2 \end{cases}$$

Raileigh quatient

$$\frac{x^T A x}{x^T x}$$

Algorithm

Given $x^0 \neq 0$.

for $i=1$, until converges.

$y_{i+1} = A x_i$

$x_{i+1} = y_{i+1} / \|y_{i+1}\|_2$

$\mu_{i+1} = x_{i+1}^T A x_{i+1}$

end

<u>Thm:</u> (i) $\lim_{k\to\infty} \mu_k = \lambda_1$, (ii) $\lim_{k\to\infty} x_k = v_1$

$A v_1 = \lambda_1 v_1$

Proof:

$$x_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|_2} = \alpha A x_k = \bar{\alpha} A^2 x_{k-1}$$

$$= \hat{\alpha} A^{k+1} x_0$$

$$= \hat{\alpha} \lambda_1^{k+1} \left(\alpha_1 v_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{k+1} v_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^{k+1} v_n \right)$$

$\to 0$ $\to 0$

$\lim_{k\to\infty} x_{k+1} = \beta v_1$

Since $\|v_1\|_2 = 1$

$\|x_{k+1}\|_2 = 1$, $\beta = 1$.

$x^T x = \|x\|_2^2$

$$\lim_{k\to\infty} \mu_k = \lim_{k\to\infty} x_k^T A x_k = \lambda_1 v_1^T v_1 = \lambda_1$$

<u>Real condition:</u> $|\lambda_1| > |\lambda_2|$

The theorem is still valid!

$x_1 \quad x_2 \cdots$

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

<u>Complexity:</u> How many matrix-vector multiplication.

$y_{i+1} = A y_i \quad O(n^2)$

$y_i^T \underline{\underline{A x_i}} \quad O(n)$

How do we find the smallest magnitute of all eigenvalues.

$A \quad \lambda_1, \lambda_2, \dots \quad \lambda_n \qquad A v_n = \lambda_n v_n.$

$$\underline{|\lambda_n|} < |\lambda_{n-1}| \leq \dots \leq |\lambda_1|$$

A^{-1}: The eigenvalues are $\qquad \det(A) \neq 0$

$$\frac{1}{\lambda_n} \quad \frac{1}{\lambda_{n-1}} \dots \quad \frac{1}{\lambda_1}$$

$$\left|\frac{1}{\lambda_n}\right| > \frac{1}{|\lambda_{n-1}|} \geq \frac{1}{|\lambda_{n-2}|} \geq \dots \geq \frac{1}{|\lambda_1|}$$

$$\begin{cases} y_{i+1} = A^{-1} x_i \implies A y_{i+1} = x_i \\ x_{i+1} = y_{i+1} / \|y_{i+1}\|_2 \\ \mu_{i+1} = x_{i+1}^T A^{-1} x_{i+1} \end{cases}$$

The inverse power method $\qquad PA = LU$

$$A y_{i+1} = x_i$$

$$\begin{cases} L U y_{i+1} = P x_i \\ x_{i+1} = y_{i+1} / \|y_{i+1}\|_2 \\ \mu_{i+1} = x_{i+1}^T \underline{\underline{A}} x_{i+1} \end{cases}$$