Direct LU decomposition (No pivoting)

SCDD (strictly column diagonally dominant)

SPD (symmetric positive definite)

Are the following SCDD, SPD

$$A = \begin{bmatrix} 5 & 0 & 1 \\ -1 & 2 & -3 \\ 3 & -1 & 7 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

A $\quad 5 > |-1| + |3|, \quad 2 > 0 + |-1|, \quad 7 > |-3| + 1$

SCDD, Not $\underline{\text{SPD}}$ symmetric part $\frac{A+A^T}{2}$. S.P.D

$\underline{\text{B:}}$ 2-nd column $2 \geq |-1| + |-1|$ $\quad \underline{\text{CDD}}$

$$C = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

SCDD ✗

SPD, ✗

Thm: If A is a SCDD, then after one step Gaussian elimination

$$A^{(1)} \longrightarrow A^{(2)} = \begin{bmatrix} a_{11} & * \\ 0 & A_1 \end{bmatrix},$$ then A_1 is also a SCDD.

$$\sum_{\substack{i=2 \\ i\neq j}}^{n} |a_{ij}^{(2)}| \overset{?}{<} |a_{jj}^{(2)}|, \quad j=2,\dots,n \qquad \begin{bmatrix} a_{11} & * \\ 0 & * \\ \vdots & \vdots \\ \vdots & a_{jj}^{(2)} \end{bmatrix}$$

$$= \sum_{\substack{i=2 \\ i\neq j}}^{n} \left| a_{ij}^{(1)} - \frac{a_{i1}^{(1)}}{a_{11}^{(1)}} a_{1j}^{(1)} \right| \qquad < \left| a_{jj}^{(1)} - \frac{a_{j1}^{(1)}}{a_{11}^{(1)}} a_{1j}^{(1)} \right|$$

$$\le \sum_{\substack{i=2 \\ i\neq j}}^{n} |a_{ij}^{(1)}| + \sum_{\substack{i=2 \\ i\neq j}}^{n} \frac{|a_{i1}^{(1)}|}{|a_{11}^{(1)}|} |a_{1j}^{(1)}|$$

$$< |a_{jj}^{(1)}| - |a_{1j}^{(1)}| + |a_{1j}^{(1)}| + \frac{|a_{j1}^{(1)}| - |a_{j1}^{(1)}|}{|a_{11}^{(1)}|}$$

??

$$\le |a_{jj}^{(1)}| - \left| \frac{a_{1j}^{(1)}}{a_{11}^{(1)}} \right| |a_{j1}^{(1)}|$$

$$|6-3| \le |6-(-3)|$$

$$= \left| a_{jj}^{(1)} - \frac{a_{j1}^{(1)}}{a_{11}^{(1)}} a_{1j}^{(1)} \right| = |a_{jj}^{(2)}|$$

$$|a| - |b| \le |a-b|$$

SPD

(i) $A = A^H$, $A \in \mathbb{C}^{n \times n}$ $a_{ij} = \overline{a_{ji}}$

(ii) for $\vec{x} \in \mathbb{C}^n$, $\vec{x} \neq 0$, $x^H A x > 0$

Conclusions: $A = LL^T$. $O(\frac{n^3}{6})$ $\times$ $\div$.

half storage

Properties of a SPD matrix.

- $\lambda_i(A) > 0$, $i = 1, 2, \ldots, n$.
- $a_{ii} > 0$, $\vec{x} = e_i$, $x^H A x = a_{ii} > 0$
- All the determinants of the principal sub-matrices are positive. (leading)

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad A_1 = [a_{11}]$$

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\det(A_i) > 0$. A_i are S.P.D.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

$\det(A_1) = \det([2]) = 2$

$\det(A_2) = \det\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)$

$= 3 > 0$

$\det(A) = 2 \cdot \det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1) \det\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$

$= 6 - 2 = 4 > 0.$

$A = A^{\frac{1}{2}} \cdot A^{\frac{1}{2}}$

$A = T^H D T$

$$A = LL^T \quad \text{or} \quad LDL^T$$

$$= \begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & 0 \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ & l_{22} & \cdots & l_{n2} \\ 0 & & \ddots & \\ & & & l_{nn} \end{bmatrix} \quad \begin{matrix} 2n \\ n2 \end{matrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad a_{ij} = a_{ji}$$

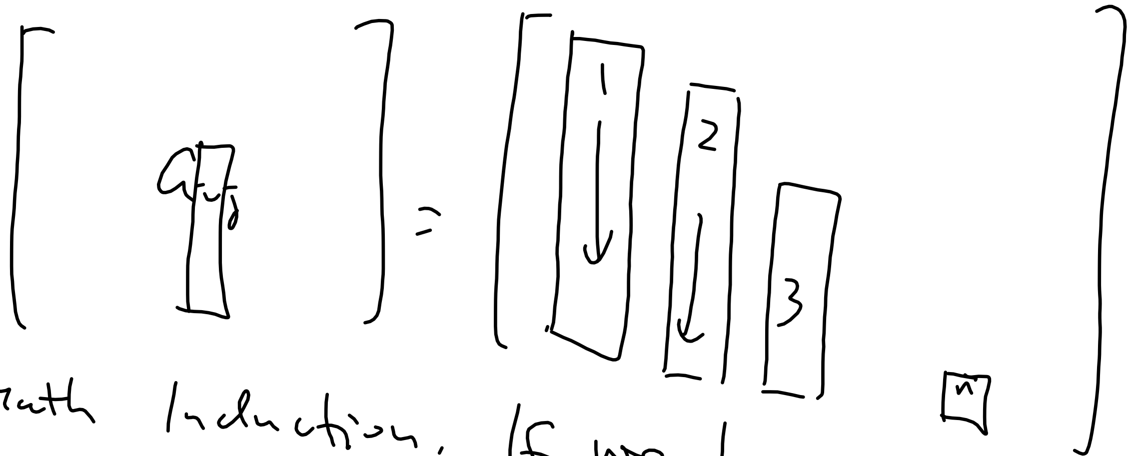
$i, j = 1, 2 \ldots n$

$$a_{ij} = \sum_{k=1}^{n} l_{ik} \overset{L^T}{u_{kj}} = \sum_{k=1}^{n} l_{ik} l_{jk}.$$

1st column. $a_{11} = l_{11}^2 \qquad l_{11} = \sqrt{a_{11}}$

$a_{i1} = l_{i1} l_{11}, \quad l_{i1} = a_{i1} / l_{11}$

ith row times 1st column



Math Induction. If we have computed 1-st, 2nd, ... ~~k~~-1 th column, try to find k-th column.

u_{jk}

(k, k)

$$a_{kk} = \sum_{j=1}^{k-1} l_{kj} l_{kj} + l_{kk}^2$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2} \longleftarrow \det(A_k)$$

Can you prove it [1]

$$a_{ik} = \sum l_{ij} u_{jk}$$

$i > k$.

$$= \sum_{j=1}^{k-1} l_{ij} l_{kj} + l_{ik} l_{kk} \checkmark$$

$$l_{ik} = \left(a_{ik} - \sum l_{ij} l_{kj}\right) / l_{kk}$$

$i = k+1, \ldots, n$.

If A is an S.P.D (SPD)

$A = LL^T$, $\quad A = LDL^T \to$ No $\sqrt{\ }$ operation

$\to$ A does not have to be SPD

$A = A^T$,

If $\det(A_i) \neq 0$. A_i's are principal submatrices of A, $i = 1, 2, \ldots n-1$.

$$\det(A_i) = \prod_{k=1}^{i-1} a_k$$

<u>Thm:</u> If A is an S.P.D matrix, then after one step GE

$$L_1 A = \begin{bmatrix} a_{11} & * \\ 0 & A_1 \end{bmatrix}$$

Then A_1 is also an S.P.D.

<u>proof</u>

$$\underline{L_1 A L_1^T} = \begin{bmatrix} a_{11} & 0 \\ 0 & A_1 \end{bmatrix}$$

$x^T A_1 x > 0$

<u>Part B:</u> $g(n) \le \sqrt{n}$ $\quad g(n) = \dfrac{\max\limits_{\substack{1\le i\le n \\ i\ge j}} |l_{ij}|}{\max\limits_{\substack{1\le i\le n \\ 1\le j\le n}} |a_{ij}|}$

For general matrices with GEPP $\quad g(n) = 2^{n-1}$

Wilkinson's conjecture: For <u>most</u> (?) matrices, $g(n) \sim n$ if GEPP is used.

Iterative methods for solving $Ax=b$.

vs Direct methods. Can get a true soln after finite steps if there are no round-off errors. GEPP

small - medium linear systems
dense

Motivations

1. For sparse matrices: The no. of non-zero entries << the no. of zeros in a matrix

Ex: Fill-ins

$$A = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 2 & 1 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 1 & 0 & \cdots & 0 \\ & & & & & \\ n & 0 & 0 & - & - & 1 \end{bmatrix}$$

No. of non-zeros
$n + (n-1)\cdot 2$
$= 2n-2$

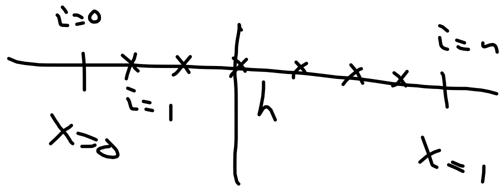
No of zeros
$n^2 - (2n-2) >> 2n-2$

GE

$$\begin{bmatrix} 1 & 1 & \cdots & & 1 \\ 0 & -1 & -2 & \cdots & -2 \\ 0 & -3 & -2 & -3 \cdots & -3 \\ \vdots & & & & \\ 0 & -n & -n & \cdots & -n+1 \end{bmatrix} \rightarrow \text{dense matrix}$$

$$\begin{cases} \dfrac{u_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i), & i = 1, 2, \ldots, n-1. \\ u_0 = 0, \quad u_n = 0 \end{cases} \longrightarrow AU = F$$

$$\begin{cases} u''(x) = f(x) & 0 \le x \le 1 \\ u(0) = 0, \quad u(1) = 0 \end{cases}$$



$$\lim_{h \to 0} \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} \approx u''(x_i) = f(x_i)$$

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i)$$

$Ax = b$, Assume $f(x)$ is known

$i = 1$, $\dfrac{\overset{0}{U_0} - 2U_1 + U_2}{h^2} = f(x_1)$

$i = 2$ $\dfrac{U_1 - 2U_2 + U_3}{h^2} = f(x_2)$

General i $\dfrac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i)$

The last one $\dfrac{U_{n-2} - 2U_{n-1} + U_n}{h^2} = f(x_{n-1})$

$Ax = b$

$$
\begin{bmatrix}
\frac{-2}{h^2} & \frac{1}{h^2} & 0 & \cdots & 0 \\
\frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & 0 \cdots & 0 \\
0 & & \ddots & & 0 \\
0 & & \ddots & \frac{1}{h^2} & \frac{-2}{h^2}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_i \\ u_{n-1} \end{bmatrix}
=
\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_i) \\ f(x_{n-1}) \end{bmatrix}
$$

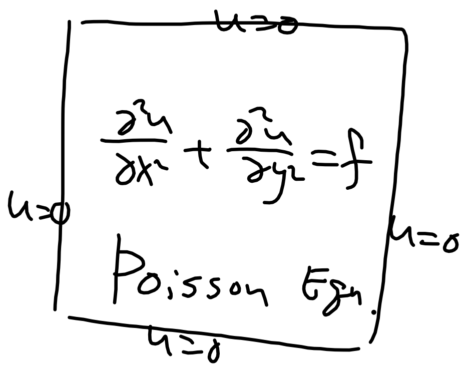
LL^T

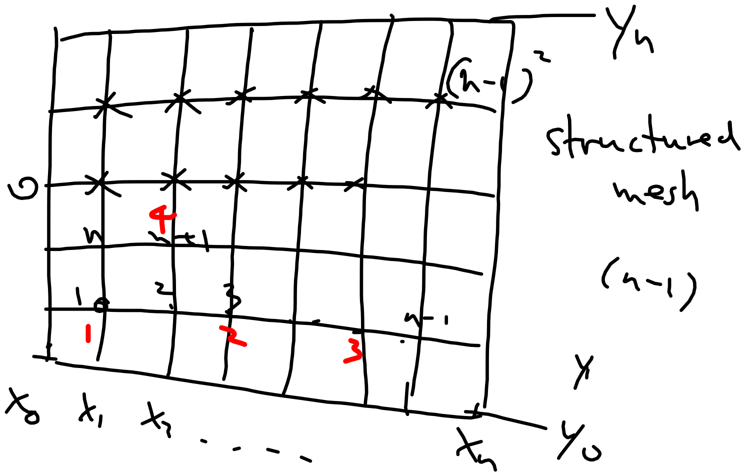
LDL^T

No. of non-zeros $(n-1) + 2(n-2) = O(3n)$

Crout decomposition $A = LU$

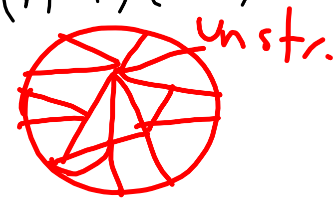
plus chase method





Fourier series

No. of unknowns $(n-1)(n-1)$



$Ax=b$ $\quad A \in \mathbb{R}^{(n-1)^2,(n-1)^2}$

How do we index the unknowns and equations: Natural ordering

Stationary iterative methods for solving $Ax=b$.

If A is large, sparse.

Other approaches:

Block QR

Sparse matrix solver

prone to errors.

$\underline{Ax=b} \iff x = Rx + c$

with the same solution $(I-R)$ should be invertible.

$x^{(k+1)} = Rx^{(k)} + c$, assume $x^{(0)}$ is an initial guess. Hope $\lim_{k\to\infty} x^{(k)}$ exists $= x^*$, $x^* = A^{-1}b$.

R is called an iteration matrix, is a constant matrix

$Ax=b$. $A - L - U$

$$A = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & & & & \\ -a_{21} & 0 & & & \\ -a_{31} & -a_{32} & \ddots & & \\ & & & \ddots & \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & 0 \end{bmatrix}$$

D

$A = D - L - U$ $\quad Ax = b$

$(D-L-U)x = b$

Splitting matrix A

$Dx = (L+U)x + b$

$x = D^{-1}(L+U)x + D^{-1}b$

$x^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b$. $\quad c = D^{-1}b$.

$x^{(k+1)} = \underline{R}x^{(k)} + c$. Jacobi iteration

Matrix-vector form. Useful analysis.

Component form. Useful for implementations.

$$U = -\begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ & 0 & -a_{23} & \cdots & -a_{2n} \\ & & \ddots & \ddots & \vdots \\ & \mathbf{0} & & 0 & -a_{n-1,n} \\ & & & & 0 \end{bmatrix}$$

$f(x) = 0$

$x = g(x)$

$x^{(k+1)} = g(x^{(k)})$

<u>1D</u> ex. $\sqrt[3]{5}$

$x = \sqrt[3]{5}$ $\quad x^3 - 5 = 0$

$x = \frac{5}{x^2}$ $\quad x^{(k+1)} = \frac{5}{(x^{(k)})^2}$, $k = 0, 1, \ldots,$ $\quad x\, x^2 - 5 = 0$

$$x^{(k+1)} = x^{(k)} - \frac{(x^{(k)})^3 - 5}{3(x^{(k)})^2},$$ Newton's method

$Ax = D.$

Jacobi iterative method $\vec{x}^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b$

The component form.

Solve for the diagonals

<u>1-st</u> $a_{11}x_1 + a_{12}x_2 \dots + a_{1n}x_n = b_1$

$$x_1 = \left(b_1 - \sum_{j=2}^{n} a_{1j}x_j\right)/a_{11}$$

$$x_2 = \left(b_2 - \sum_{\substack{j=1\\ j\neq 2}}^{n} a_{2j}x_j\right)/a_{22}$$

$$x_i = \left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j\right)/a_{ii}$$

Jacobi iterative method: Given an initial guess, $x_i^{(k+1)} = \left(b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right)/a_{ii}$, $i = 1, 2, \dots n.$

$k = 1, 2, \dots$ until converges. Only need matrix-vector multiplication.

Stopping Criteria: tol: tolerance

10^{-6}, 10^{-8}

1. $\|\vec{x}^{(k+1)} - \vec{x}^{(k)}\| \le tol$

2. $\|A\vec{x}^{(k)} - b\| \le tol$

3. $\dfrac{\|\vec{x}^{(k+1)} - \vec{x}^{(k)}\|}{\|\vec{x}^{(k)}\|} \le tol$

4. $k \le k_{max}$, $k_{max} = 10^5$

Pseudo - Code

$\vec{x}^{(1)}, \vec{x}^{(2)} \ldots \vec{x}^{(k)}, .$

We need just $\vec{x}^{(k)}, \vec{x}^{(k+1)}$

$\vec{x}^{(k+1)}$: $x^{(new)}$

$\vec{x}^{(k)}$: $\cancel{x}^{(old)}$

x1, x2

Inputs: x, tol %x(n)

1e10.

```
error = 1000;   xnew = x0;
while error > tol              x = b - Σ
for i = 1 : n
    xnew(i) = b(i);
                                  . neg.
    for j = 1 : n
        if j ≠ i,                  j < > i
            xnew(i) = xnew(i) - a(i,j) * x(j);
        end
    end
    xnew(i) = xnew(i)/a(i,i);
end
error = norm(xnew - x);
  x = xnew;
end      % end of While !
```

one-step Jacobi

Ex: $A=\begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix}$, $b=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{x}^{(0)}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Jacobi:

$\vec{x}^{*}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x_1^{(k+1)} = \left(1 + 9x_2^{(k)}\right)/10$$

$$x_2^{(k+1)} = \left(1 + 9x_1^{(k)}\right)/10.$$

$x_1^{(0)}=0$ $\quad x_1^{(1)} = 1/10,$

$x_2^{(0)}=0$ $\quad x_2^{(1)} = 1/10.$

$$x_1^{(2)} = \left(1 + 9\cdot\frac{1}{10}\right)/10$$

$$x_2^{(2)} = \left(1 + 9/10\right)/10.$$

Gauss-Seidel iterative method

(Use the most updated information)

$$x_1^{(k+1)} = \left(b_1 - \sum_{j=2}^{n} a_{1j} x_j^{(k)}\right) / a_{11}$$

$$x_2^{(k+1)} = \left(b_2 - a_{21} x_1^{(k+1)} - \sum_{j=3}^{n} a_{2j} x_j^{(k)}\right) / a_{22}$$

$$x_i^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right) / a_{ii}$$

Matrix-Vector form

$$(D-L)x = Ux + b$$

$$\vec{x}^{(k+1)} = \underline{(D-L)^{-1} U} x^{(k)} + (D-L)^{-1} b$$

<u>Jacobi</u> $Dx = (L+U)x + b$

$$x^{(k+1)} = (D^{-1}(L+U))\vec{x}^{(k)} + D^{-1}b$$

$$R_J = D^{-1}(L+U), \quad C_J = D^{-1} b$$

$$R_{G\text{-}S} = (D-L)^{-1} U, \quad C_{G\text{-}S} = (D-L)^{-1} b.$$