Error from GEPP

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)}$$

$$fl(a_{ij}^{(k+1)}) = \left(a_{ij}^{(k)}(1+\delta_1) - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}(1+\delta_1) a_{kj}^{(k)} (1+\delta_3)\right)(1+\delta_4)$$

$$= a_{ij}^{(k)} \delta_1 + \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)} \delta$$

Define the growth factor

$$g^{(n)} = \frac{\max_{1 \le k \le n-1} \max_{k \le i,j \le n} |a_{ij}^{(k)}|}{\max_{1 \le i,j \le n} |a_{ij}^{(1)}|}$$

Estimate the $g(n)$

$$|a_{ij}^{(k+1)}| = \left| a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)} \right|$$

$$\leq |a_{ij}^{(k)}| + \underline{\left| \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \right|} |a_{kj}^{(k)}|$$

$$\leq |a_{ij}^{(k)}| + |a_{kj}^{(k)}|$$

$$\leq 2 \max_{k \leq i,j \leq n} |a_{ij}^{(k)}| \leq 2 \cdot 2 \max |a_{ij}^{(k-1)}|$$

$$\leq$$

$g(n) \leq 2^{n-1}$ The $\underline{\underline{\text{optimal}}}$ upper bound!

$$A = \begin{bmatrix} 1 & & & & 1 \\ -1 & 1 & & 0 & 1 \\ -1 & -1 & 1 & & 1 \\ & \cdots & & & 1 \\ -1 & -1 & -1 & \cdots & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \cdots & & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 2 \\ 0 & -1 & +1 & \cdots & 0 & 2 \\ 0 & -1 & -1 & 1 & \cdots & 2 \\ \vdots & & & & & \\ 0 & -1 & -1 & \cdots & 1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 2 \\ 0 & 0 & 1 & \cdots & 0 & 4 \\ & & & & & \end{bmatrix} \rightarrow \begin{bmatrix} 1 & & & & 1 \\ & 1 & & & 2 \\ & & \ddots & & 4 \\ & & & \ddots & 8 \\ & & & & 2^{n-1} \end{bmatrix}$$

$$\max_{1 \le i,j \le n} |a_{ij}| = 1 \qquad \max_{1 \le i,j \le n} \left| a_{ij}^{(n-1)} \right| = 2^{n-1}$$

Error bound for GEPP.

<u>Thm:</u> Let $A \in \mathbb{R}^{n\times n}$, $\det(A) \neq 0$, $b \in \mathbb{R}^{n\times 1}$, $\|A^{-1}\|_p \leq$
$\|A^{-1}\delta A\| \leq \|A^{-1}\|_p \|A\|_p \varepsilon < 1$. $\bar{X}$ is the computed solution using GEPP, then

$$\frac{\|X_e - \bar{X}\|}{\|X_e\|} \leq \underbrace{\varepsilon}_{\text{computer}}\ \underbrace{\mathrm{cond}(A)}_{\text{problem}}\ \underbrace{g(n)}_{\text{algorithm}} + O\left((\mathrm{cond}(A)\,\varepsilon)^2\right)$$

How do we find Cond(A),

$\|A\|_p$, $p = 1, \infty$ Difficult to get $\|A^{-1}\|$

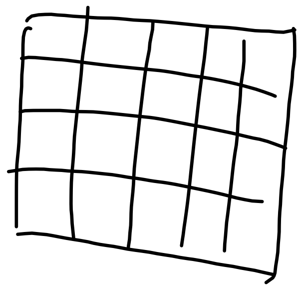
Solve $A^{-1}b$, $\det(A)$, $\|A^{-1}\|$ have same difficulties.

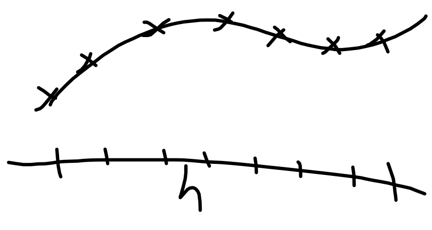
We can get estimates of $\|A^{-1}\|$ from: analyze the problem, algorithms.

Finite difference / element methods to solve ODE / PDE

Ex 1

$\mathrm{Cond}(A) \sim \frac{1}{h^2}$ $\quad -u''(x) = f(x)$ mesh size h





Ex 2.

Hilbert matrix

$$\left\{ -\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i) \right.$$

$i = 1, 2, \ldots n-1$

Find the smallest eigenvalues

$$\mathrm{Cond}_2(A) = \frac{\sigma_{max}}{\sigma_{min}} \qquad \sigma_{min} = \min_{1 \le i \le n} \sqrt{\underline{\lambda_i(A^TA)}}$$

Using the inverse Power method.

Use the residual vector to estimate how close $\bar{x}$ to x_e.

Give a vector, $\bar{x}$

$r(x) = b - A\bar{x}$ Computable

A^{-1}

$r(x) = 0$, $\bar{x} = x_e$ cost of $A\bar{x}$

$O(n^2)$

$\|r(\bar{x})\| \to 0$ $\bar{x} \to x_e$

cond(A)

$\frac{\|\bar{x} - x_e\|}{\|x_e\|} \leq \|r(\bar{x})\| \leq \frac{\|\bar{x} - x_e\|}{\|x_e\|}$

cond(A)

Thm: $$\frac{\|r(\bar{x})\|}{\|A\|\,\|x_e\|} \le \frac{\|\bar{x}-x_e\|}{\|x_e\|} \le \underline{\|A^{-1}\|}\,\frac{\|r(\bar{x})\|}{\|x_e\|}$$

$$\le \|r(\bar{x})\| \le$$

Proof one:

$$\frac{\|x_e-\bar{x}\|}{\|x_e\|} = \frac{\|A^{-1}b-\bar{x}\|}{\|x_e\|} = \frac{\|A^{-1}\overset{r(\bar{x})}{(b-A\bar{x})}\|}{\|x_e\|}$$

$$\le \frac{\|A^{-1}\|\,\|r(\bar{x})\|}{\|x_e\|}$$

$$\|r(\bar{x})\| = \|b-A\bar{x}\| = \|A(\overset{x_e}{\underline{A^{-1}b}}-\bar{x})\|$$

$$\le \|A\|\,\|x_e-\bar{x}\|$$

$$\frac{\|r(\bar{x})\|}{\|A\|} \le \|x_e-\bar{x}\|$$

Direct LU decomposition $PA = LU$

Not always possible $A = LU$

For special matrices

Band matrix, e.g. tridiagonal

strictly column diagonally dominant

Symmetric positive definite (SPD)

Applications: Theoretical, fast solvers, preconditioning e.g. Incomplete LU

$\text{Cond}(I) = 1$ $\text{cond}(BA) \sim O(1)$

$\text{Cond}(A) \geq 1$ preconditioning

Can you prove it? $A \sim LU$

$A^{-1} \sim U^{-1} L^{-1}$

$$A = LU = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & 0 \\ l_{31} & l_{32} & \ddots & \\ l_{n1} & l_{n2} & \cdots l_{n\,n-1} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \\ 0 & & & u_{nn} \end{bmatrix}$$

Matrix Eqn. Derivation, Math induction

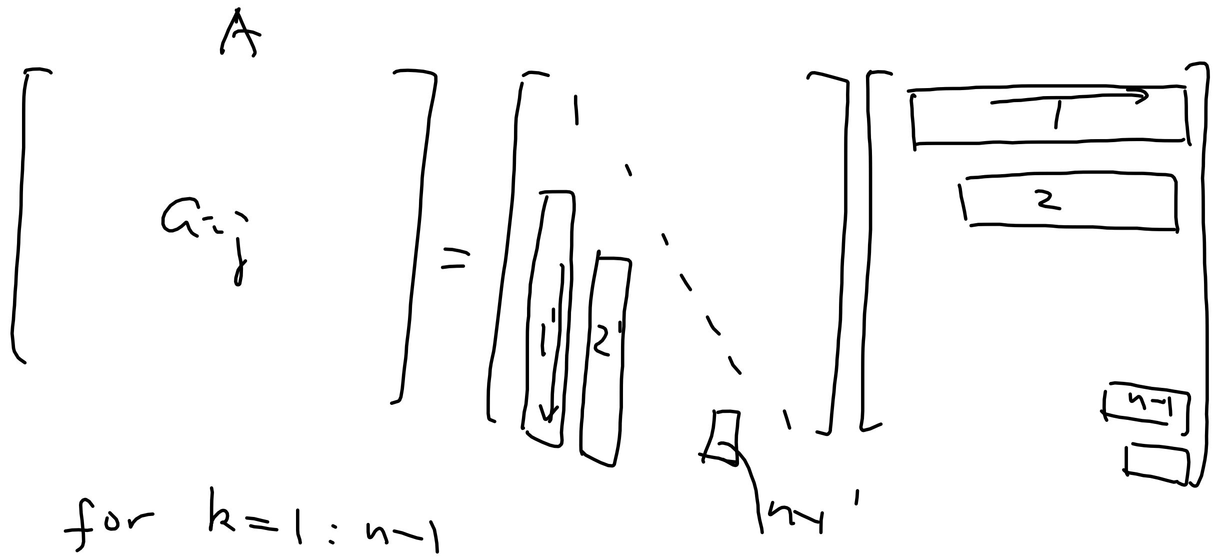
$a_{1j} = 1\, u_{1j}$ $\quad u_{1j} = a_{1j}, \; j = 1, 2 \ldots n$

$l_{i1} u_{11} = a_{i1}$ $\quad l_{i1} = a_{i1} / a_{11} \quad i = 2, \ldots, n$

$l_{21} u_{12} + 1 \cdot u_{22} = a_{22}$ $\quad u_{22} = a_{22} - l_{21} u_{12}$

$l_{21} u_{1j} + u_{2j} = a_{2j}$ $\quad u_{2j} = a_{2j} - l_{21} u_{1j}$

$j = 2, \ldots n$



for $k = 1 : n-1$

  for $j = k : n$

$$u_{kj} = a_{kj} - \sum_{i=1}^{k-1} l_{ki} u_{ij}$$

  end

  for $i = k+1 : n$

$$l_{ik} = \left(a_{ik} - \sum_{j=1}^{k-1} l_{ij} u_{jk}\right) / u_{kk}$$

  end

end

Computational cost $O(n^3/3)$ multiplications / divisions

$$A = \begin{bmatrix} d_1 & \beta_1 & & 0 \\ \alpha_2 & d_2 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ 0 & & \alpha_n & d_n \end{bmatrix}$$

Appl. ADI

1D BVP

$$= \begin{bmatrix} 1 & & & \\ \alpha_2' & 1 & & 0 \\ & \ddots & \ddots & \\ & & \alpha_n' & 1 \end{bmatrix} \begin{bmatrix} d_1' & \beta_1 & & \\ & d_2' & \beta_2 & \\ & 0 & \ddots & \beta_{n-1} \\ & & & d_n' \end{bmatrix}$$

$d_1' = d_1$, $\alpha_2' = \alpha_2 / d_1'$

For general i $\quad \alpha_i \beta_{i-1} + d_i' = d_i$

$\checkmark \; d_i' = d_i - \alpha_i' \beta_{i-1} \quad i = 1, 2, \ldots, n$

$\checkmark \; \alpha_i' = \alpha_i / d_{i-1}'$

Complexity $\times / \div$

Crout algorithm $O(2n)$ $\qquad O(n^3/3)$

$A = LU$

$Ax = b$

$L\underbrace{Ux}_{y} = b$

$\begin{cases} Ly = b \\ Ux = y \end{cases}$

$O(5n)$

$O(n)$

$Ux = y$

$O(2n)$

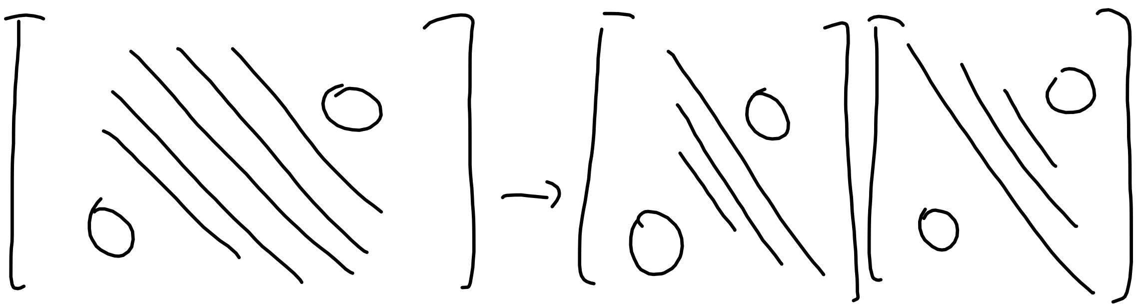
$$\begin{bmatrix} 1 & & & \\ \alpha_2' & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \alpha_n' & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$y_1 = b_1$

$\alpha_i' y_{i-1} + y_i = b_i$

$y_i = b_i - \underline{\alpha_i' y_{i-1}}$

$i = 2, \ldots, n.$



The structure remains unchanges

Strictly column diagonally dominant matrix

$$|a_{11}| > \sum_{\substack{i=2\\ i\neq 1}}^{n} |a_{i1}|$$

$$|a_{ii}| > \sum_{\substack{k=1\\ k\neq i}}^{n} |a_{ki}| \to \sum_{k=1}^{i-1} |a_{ki}| + \sum_{k=i+1}^{n} |a_{ki}|$$

Thm. If A is SCDD, then after one step GE

$$LA = \begin{bmatrix} a_{11} & * \\ 0 & A_1 \end{bmatrix}$$

Then A_1 is also SCDD.