How to exchange rows of a matrix

permutation matrix

$$P_{ij} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix}$$

$$I = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \end{bmatrix} \begin{matrix} i \\ \\ j \end{matrix}$$

an elementary permutation matrix

$\det(P_{ij}) = -1$

P_{ij}^{-1} exists.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(P_{ij})\ P_{ij} A = \begin{matrix} i \to \\ j \to \end{matrix} \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \det(P_{ij}) =$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ & & \\ \to\ a_{j1} & a_{j2} & \cdots\ a_{jn} \\ & & \\ \to\ a_{i1} & a_{i2} & \cdots\ a_{in} \end{bmatrix}$$

$P_{ij}^{-1} = P_{ij}$

$P_{ij}^{T} = P_{ij}$

Orthogonal matrix.

Instead of $A=LU$ (not always possible)

We should use $PA=LU$

L_{n-1}

$P_{n-2}L_2P_{2,i2}\underline{L_1P_{1,i1}A} = U$ an upper triangular

Ex.

$$P=P_{13}\quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&2&3\\-2&4&5\\3&-1&0\end{bmatrix}$$

$$=\begin{bmatrix}3&-1&0\\-2&4&5\\1&2&3\end{bmatrix}$$

$P_{21}\underline{L_1P_{11}A} =$

$$P_{24}L_1 = \underbrace{\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}1&0&0&0\\-\frac{1}{3}&1&0&0\\\frac{2}{3}&0&1&0\\1&0&0&1\end{bmatrix}}_{L}$$

$$=\tilde{L}_1P_{24} = \begin{bmatrix}1&0&0&0\\1&0&0&1\\\frac{2}{3}&0&1&0\\-\frac{1}{3}&0&0&1\end{bmatrix}$$

Thm: After Gaussian elimination with partial column pivoting, we have

$$PA = LU$$

$P = P_{n-2,i_{n-2}} P_{n-1,i_{n-1}} \cdots P_{1,i_1}$

L is a unit triangular matrix

$\det(A) = (-1)^m \det(U)$

m is the number of row exchanges.

Ex: $A = \begin{bmatrix} 1 & -1 & 6 \\ 2 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix}$.

Get ① $PA = LU$

② Solve $Ax = b$.

$$\begin{bmatrix} 1 & -1 & 6 \\ 2 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -1 & 6 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{matrix} -1 & 0 & -1 \\ 1 & -1 & 6 \end{matrix}$$

$$\begin{matrix} -1 & 0 & -1 \\ 1 & 2 & 4 \end{matrix}$$

$$\xrightarrow{L_1} \begin{bmatrix} 2 & 0 & 2 \\ \frac{1}{2} & -1 & 5 \\ \frac{1}{2} & 2 & 3 \end{bmatrix} \xrightarrow{P_{23}} \begin{bmatrix} 2 & 0 & 2 \\ \frac{1}{2} & 2 & 3 \\ \frac{-1}{2} & -1 & 5 \end{bmatrix}$$

$$\xrightarrow{L_2} \begin{bmatrix} 2 & 0 & 2 \\ \frac{1}{2} & 2 & 3 \\ \frac{1}{2} & \frac{-1}{2} & \frac{13}{2} \end{bmatrix}$$

$$\det(A) = (-1)^2 \cdot 2 \cdot 2 \cdot \frac{13}{2} = 26$$

$$U = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 13/2 \end{bmatrix}. \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

$P = P_{23} P_{12}$

$Ax = b$

$PAx = Pb$

$LUx = Pb$

$Ly = Pb$

$Ux = y$

$P_{23} P_{12} b$ $\quad b = \begin{bmatrix} 6 \\ 4 \\ 7 \end{bmatrix}$

$= P_{23} \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} y = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$

$y_1 = 4$

$y_2 = 7 - \frac{1}{2} \cdot 4 = 5$

$y_3 = 6 - \frac{1}{2} y_1 + \frac{1}{2} y_2 = \frac{13}{2}$

$Ux = y$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{13}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ \frac{13}{2} \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Other pivoting techniques.

1. Complete pivoting $\max_{1 \le i \le n, 1 \le j \le n} |a_{ij}^{(1)}|$

   Unnecessarily complicated.

2. Scaled pivoting (useful)

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 100 & 200 & -300 & 400 \end{bmatrix} \xrightarrow{\text{scale}} \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & \\ 1 & 2 & -3 & 8 & * \\ & & & & \end{array}\right]$$

$$\frac{|a_{k1}|}{s_k} = \max_{1 \le i \le n} \frac{|a_{i1}|}{s_i} \qquad s_i = \sum_{j=1}^{n} |a_{ij}|$$

Factors affect accuracy ① algorithm growth factor $g(n)$
② machine precision
③ condition of the problem

$x \cdot y$ Well conditioned

$x - y$ may be ill-conditioned if $x \sim y$.

Error analysis for solving $Ax = b$

$fl(b_i) = b_i(1+\delta_i)$, $|\delta_i| \le \varepsilon$

$fl(\vec{b}) = \vec{b} + \delta\vec{b}$

$\|\delta b\|_p = \left\| \begin{bmatrix} \delta b_1 \\ \delta b_2 \\ \vdots \\ \delta b_n \end{bmatrix} \right\|_p$

$|\delta b_i| \le |b_i|\varepsilon$ $\le |b_1|\varepsilon$

$\|\delta b\|_p \le \left\| \begin{matrix} |b_1|\varepsilon \\ \vdots \\ |b_n|\varepsilon \end{matrix} \right\|_p = \varepsilon \||b|\|_p = \|b\|_p \varepsilon$

$p = 1, 2, \infty$

$\|\delta b\| \le C\|b\|\varepsilon$

$fl(A) = A + \delta A$ $\quad\quad$ $\|\,|B|\,\|_p = \|B\|_p$, $p = 1, 2, \infty$

Actually we are solving $(A+\delta A)x = b + \delta b$ $\quad\quad$ $\sup \frac{\|Ax\|}{\|x\|}$

If $x^* = A^{-1}b$, $\quad \bar{x} = (A+\delta A)^{-1}(b+\delta b)$

$$\frac{\|x^* - \bar{x}\|}{\|x^*\|} \leq \|A\|\,\|A^{-1}\| \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right)$$

Input error

$$= \mathrm{Cond}(A) \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right) \leq$$

Sensitivity

If cond(A) is big, then it is sensitive, ill-conditioned

Otherwise it is called well-conditioned

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \cdots + (-1)^n x^n \cdots$$

$|x| < 1$ $= \lim_{n\to\infty} \frac{1-x^n}{1+x} = \frac{1}{1+x}$

Banach Thm.

└ partial sum

$(1+x)^q = 1 + qx$

$$(A + \delta A)^{-1} = \left(A(I + A^{-1}\delta A)\right)^{-1}$$

$$= (I + A^{-1}\delta A)^{-1} A^{-1}$$

$$= (I + E)^{-1} A^{-1}$$

Lemma. If $\|E\| < 1$, then $I+E$ is invertible, and

$$\|(I+E)^{-1}\| \leq \frac{1}{1-\|E\|}$$

Banach's Theorem.

$x_e = A^{-1}b$, $\quad (A+\delta A)x = b+\delta b$

$x_e \neq \vec{0}$ $\quad \bar{x} = (A+\delta A)^{-1}(b+\delta b)$

Main theorem.

If $\|A^{-1}\delta A\| < 1$, then $A+\delta A$ is invertible and $\quad \text{cond}(A) = \|A\|\,\|A^{-1}\|$

$\kappa(A) = \|A\|_2\,\|A^{-1}\|_2$

$$\frac{\|x_e - \bar{x}\|}{\|x_e\|} \leq C\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)\text{cond}(A)$$

In particular. GEPP $\quad A + \delta A_1 + \delta A_2$ — input error from GEPP

$$\frac{\|x_e - \bar{x}\|}{\|x_e\|} \leq C\, g(n)\,\text{cond}(A)\,\varepsilon$$

Banach's Theorem — growth factor

Worst

$g(n) = 2^{n-1}$

$\sim n$

$\|E\|<1$, $(I+E)^{-1}$ exists. $\frac{1}{1+x} = 1-x$

$$\|(I+E)^{-1}\| \le \frac{1}{1-\|E\|}$$

Proof

$$B = I - E + E^2 - E^3 + E^4 + (-1)E^n$$

$$B_n = I - E + E^2 \dots + (-1)E^n$$

$$(I+E)B_n = (I+E)(I - E + E^n \dots + (-1)^n E^n)$$

$$= I - E + E \dots$$

$$= I - (-1)^n E^{n+1} \to I$$

$$\|(-1)^n E^{n+1}\| = \|E^n \, E\| \le \|E^n\| \|E\| \le \|E^{n-1} \cdot E\|$$

$$\lim_{n\to\infty} B_n = (I+E)^{-1} \qquad \le \|E\|^{n+1} \to 0$$

$$\|(I+E)^{-1}\| = \| I - E + E^2 \dots \|$$

$$\le \|I\| + \|E\| + \|E^2\| \dots \|E^n\|$$

$$\le 1 + \|E\| + \|E\|^2 \dots \|E\|^n \dots$$

$$= \frac{1}{1-\|E\|}$$

$Ax=b, \quad x_e=A^{-1}b$

$(A+\delta A)x=b+\delta b, \quad \bar{x}=(A+\delta A)^{-1}(b+\delta b)$

$$\|\delta x\| = \|\bar{x}-x_e\| = \|(A+\delta A)^{-1}(b+\delta b) - A^{-1}b\|$$

$$= \|(A+\delta A)^{-1}(b+\delta b-(A+\delta A)A^{-1}b\|$$

$$\le \|(A+\delta A)^{-1}\|\,\|b+\delta b-b-\delta A x_e\|$$

$$\le \|(A(I+A^{-1}\delta A))^{-1}\|\left(\|\delta b\|+\|\delta A\|\,\|x_e\|\right)$$

$$\le \|(I+A^{-1}\delta A)^{-1}\|\,\|A^{-1}\|\left(\|\delta b\|+\|\delta A\|\,\|x_e\|\right)$$

If ① $\|A^{-1}\delta A\|<1$, then $A+\delta A$ is invertible

② $\|A^{-1}\delta A\| \le \|A^{-1}\|\,\|\delta A\|$, this is a stronger condition used by many books.

Ex: Hilbert matrix.

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & & \\ & & & \\ & & \frac{1}{i+j} & \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

H $H^T = H$. symmetric positive definite

H^{-1} exists $\lambda_i(H_n) > 0$ SPD $A = LL^T$

Choleski

$$\|\delta x\| \le \frac{\|A^{-1}\|}{1-\|A^{-1}\delta A\|\|} \left(\|\delta b\| + \|\delta A\|\,\|x_e\|\right)$$

$$\le \frac{\|A^{-1}\|\,\|A\|}{1-\|A^{-1}\|\|\delta A\|} \left(\frac{\|\delta b\|}{\|A\|} + \frac{\|\delta A\|}{\|A\|}\,\|x_e\|\right)$$

$$\frac{\|\delta x\|}{\|x_e\|} \le \|A^{-1}\|\,\|A\| \cdot \left(\frac{\|\delta b\|}{\|A\|\,\|x_e\|} + \frac{\|\delta A\|}{\|A\|}\right) \Big/ \left(1-\|A^{-1}\|\,\|\delta A\|\right)$$

$\frac{1}{1-\varepsilon} = 1+\varepsilon+O(\varepsilon)$

$Ax_e = b$

$\|b\| \le \|A\|\,\|x_e\|$

$$\le \text{cond}(A)\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right) + O\left(\|\delta A\|\,\|\delta b\| + \|\delta A\|^2\right)$$