$$\|A\|_\infty = \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} \qquad A \in \mathbb{R}^{n \times n}.$$

$$M = \sum_{j=1}^{n} |a_{i_0, j}| \qquad \|Ax\|_\infty \le M$$

$$\|x\|_\infty = 1$$

$$\max_{1 \le i \le n} |x_i| = 1$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

sign

$$x^* = \begin{bmatrix} \mathrm{sgn}(a_{i_0,1}) \\ \mathrm{sgn}(a_{i_0,2}) \\ \vdots \\ \mathrm{sgn}(a_{i_0,n}) \end{bmatrix}$$

$$\begin{bmatrix} \\ a_{i_0,1} \cdots \\ \\ \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \\ \end{bmatrix} \quad x^*$$

$$Ax^* = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \times \\ \times \\ \sum_{j=1}^{n} |a_{i_0,j}| \\ \times \\ \vdots \end{bmatrix} \end{array} \begin{array}{l} \\ \\ i_0 \\ \\ \\ \end{array}$$

$$\|Ax^*\|_\infty = M. \quad \Rightarrow \quad \|A\|_\infty = M = \max_{1 \le i \le n} \left\{ \sum |a_{ij}| \right\}$$

computable

$\|A\|_2$, $\|A\|_2 = \sqrt{\max\limits_{1\le i\le n} (A^T A)}$

$A^T A$ is a symmetric positive semi-definite matrix: $\forall\, x \in \mathbb{R}^n$, $\underline{x^T A^T A x} \geq 0$

$(Ax)^T Ax = \|Ax\|_2^2$ $\quad \|x\|_2^2 = \sum_{i=1}^n x_i^2$

An inner product with itself $= \underline{x^T x}$

All eigenvalue of $A^T A$ are non-negative,

$A = U \Sigma V^H$ $\quad \lambda_i(A^T A) \geq 0$

λ $\quad A^T A x_0 = \underline{\lambda(A^T A)}\, x_0$, $\quad x_0^T x_0 = 1$, $\|x\|_2^2 = 1$

$x_0^T A^T A x_0 = \lambda\, x_0^T x_0 = \lambda$

$\|Ax_0\|_2^2 = \lambda \geq 0$

Using these, we can show that

$\|A\|_2 = \sqrt{\max\limits_{1\le i\le n} \{\lambda_i(A^T A)\}}$

Ex: $A = \begin{array}{cc} \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} & \begin{array}{c} 4 \\ 4 \end{array} \\ \begin{array}{cc} 4 & 4 \end{array} & \end{array}$, Find $\|A\|_p$, $p = 1, 2, \infty$

$\|A\|_1 = \|A\|_\infty = 4$ → skew-symmetric

S.P.D

$$A^T A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 16$, $\|A\|_2 = \sqrt{16} = 4$

Ex 2: $A = \begin{array}{cc} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} & \begin{array}{c} 2 \\ 1 \end{array} \\ \begin{array}{cc} 1 & 2 \end{array} & \end{array}$

$\|A\|_\infty = 2$

$\|A\|_1 = 2$

$\|A\|_2$

eig(A)

$$A^T A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

AA^T

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 2 \end{pmatrix} = 0$$

$$(\lambda - 1)(\lambda - 2) - 1 = 0$$

$$\lambda^2 - 3\lambda + 2 - 1 = 0 \qquad \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_1 = \frac{3 + \sqrt{9 - 4}}{2} = \frac{3 + \sqrt{5}}{2}$$ the larger one!

$$\|A\|_2 = \sqrt{\frac{3 + \sqrt{5}}{2}}$$

$$\frac{f(x)}{x \in D,} \Rightarrow \|f\| = \sup_{\substack{x \neq 0 \\ x \in D.}} \frac{|f(x)|}{|x|}$$

$$\vec{f}(\vec{x}) \Rightarrow \|\vec{f}\| = \sup_{\substack{x \neq 0 \\ x \in D}} \frac{\|\vec{f}(\vec{x})\|}{\|x\|} \neq \max$$

Operator norms.

Solving a linear system of equations

$Ax=b$ using direct methods.

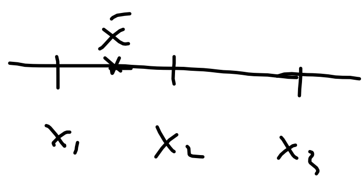
Direct Methods vs iterative

With exact arithmatics, we can get the exact soln $A^{-1}b$ in finite steps.

$Ax=b$, $A\in\mathbb{R}^{n\times n}$, $\det(A)\neq 0$, A^{-1} exists and it's unique.

$x_e = A^{-1}b$, e.g.: Interpolation

<u>Ex</u> $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$



$$x_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

$$x_2 = \frac{\det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

$$u(\bar{x}) = \alpha_1 u(x_1) + \alpha_2 u(x_2) + \alpha_3 u(x_3)$$

Computing the determinant is considered as the same complexity as find A^{-1}.

$Ax=b$, $\Leftrightarrow \det(A)$,

A^{-1} (more difficult)

$\mathrm{Cond}(A) = \|A\|\,\|A^{-1}\|$

Observation: It is easy to solve a (an) upper / lower triangular system.

Upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ 0 & & \ddots & \\ & & & a_{nn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$x_n = b_n / a_{nn}$ $\qquad \det(A) = \prod_{i=1}^{n} a_{ii}$

$x_{n-1} = (b_n - a_{n-1,n} x_n) / a_{n-1,n-1}$ $\qquad a_{n-1,n-1} x_{n-1} + a_{n-1,n} x_n = b_n$

$\vdots$

$x_i = \left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j\right) / a_{ii}$ $\qquad \sum_{j=i}^{n} a_{ij} x_j = b_i$

Backward substitution $\qquad i = n, n-1, \ldots 1$

In matlab $\qquad x = A \backslash b$

```
for i=n, -1, 1
   x(i)=b(i);
  for j=i+1:n
   x(i)= x(i) - a(i,j)* x(j);
  end
   x(i)=x(i)/a(i,i);
end
```

Complexity analysis: how any $+, -, \times, \div$

x_i: $\approx n-i$, $\quad i = 1, 2, \ldots \quad \underline{\sum a_{ij} x_j}$ $\quad i+1, \ldots n \quad n-i$

$\sum_{i=1}^{n} (n-i) = 1 + 2 + \cdots + n$

$= \frac{n}{2}(n+1) = O\left(\frac{n^2}{2}\right)$ $\qquad \times \quad \div$

Gaussian elimination (GE) for solving $Ax = b$, $A \in \mathbb{R}^{n \times n}$, $\det(A) \neq 0$?

The augmented system

$$[A \mid b] \xrightarrow[\text{the same soln.}]{\text{Transformation}} Ax = b \qquad \text{row transformation}$$

$$\left[\begin{array}{cccc|c} * & * & * & 1 & 1 \\ \vert & \vert & \vert & \vert & \vert \\ \vert & \vert & \vert & \vert & \vert \\ \vert & \vert & \vert & \vert & \vert \end{array}\right] \longrightarrow \left[\begin{array}{cccc|} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{array}\right] \quad \text{indexing}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right]$$

for $k = 1, n-1$ $\qquad (n-1) \to$ Triangular

end

How ?

$$\begin{bmatrix} 1 & & & \\ y & 1 & & 0 \\ -\frac{a_{31}}{a_{11}} & & \ddots & \\ -\frac{a_{i1}}{a_{11}} & & & 1 \end{bmatrix} \overset{A^{(1)}}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots \boxed{\cdot} \cdots & a_{2n} \\ & & & \\ a_{n1} & & & a_{nn} \end{bmatrix}} \overset{b}{\begin{bmatrix} a_{1,n+1} \\ a_{2,n+1} \\ \\ a_{n,n+1} \end{bmatrix}}$$

$-\frac{a_{n1}}{a_{11}}$

2nd row × 1-st colm

$$y a_{11} + a_{21} = 0$$

$$y = -\frac{a_{21}}{a_{11}}$$

$$= \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & & & & \\ 0 & & a_{ij}^{(2)} & & a_{i,n+1}^{(2)} \\ \vdots & & & & \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & a_{1,n+1}^{(1)} \\ 0 & & & & \\ & & & & \\ & & & & \end{array}\right]$$

If $a_{11} \neq 0$, then we choose L_1

$$a_{2j}^{(2)} = y a_{1j}^{(1)} + a_{2j}^{(1)} = -\frac{a_{21}^{(1)}}{a_{11}^{(1)}} a_{1j}^{(1)} + a_{2j}^{(1)}, \quad j = 2, \ldots, n+1$$

$$= a_{2j}^{(1)} - \frac{a_{21}^{(1)} a_{1j}^{(1)}}{a_{11}^{(1)}}$$

$$L_1 = \begin{bmatrix} 1 & & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & 0 & \\ -\frac{a_{i1}}{a_{11}} & & \ddots & & \\ & 0 & & \ddots & \\ -\frac{a_{n1}}{a_{11}} & & & & 1 \end{bmatrix}$$

$$L_1 [\overset{A^{(1)}}{A} \mid b] = \left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12} & \cdots & a_{1n}^{(1)} & a_{1n+1}^{(1)} \\ 0 & & & & \\ 0 & & a_{ij}^{(2)} & & a_{in+1}^{(2)} \\ \vdots & & & & \\ 0 & & & & \end{array}\right]$$

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i1}^{(1)}}{a_{11}^{(1)}} a_{1j}^{(1)}$$

$$A^{(1)} \to A^{(2)} \to A^{(3)}$$

for $i = k+1, n$

for $j = k+1, n+1$

for $k=1, n-1$
  for $i=k+1:n$
    for $j=k+1, n+1$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)}$$

    end
  end
end

$$A^{(1)} \to A^{(2)} \cdots \to A^{(n)}$$

$$= [A^{(n)} \mid b^{(n)}] \to \text{triangular solver}$$

Analysis. Does it work? $a_{kk}^{(k)} \neq 0$.

$$L^{(1)} A^{(1)} = A^{(2)}, \qquad L^{(1)} = \begin{bmatrix} 1 & & & \\ -\frac{a_{21}^{(1)}}{a_{11}^{(1)}} & 1 & & 0 \\ -\frac{a_{i1}^{(1)}}{a_{11}^{(1)}} & & \ddots & \\ -\frac{a_{n1}^{(1)}}{a_{11}^{(1)}} & 0 & & 1 \end{bmatrix}$$

$\det(L^{(1)}) = 1$, $\det(L^{(1)} A^{(1)}) = \det(A^{(2)})$,

$\det(L^{(1)}) \det(A^{(1)}) = \det(A^{(2)})$

...

$$\det(A) = \prod_{k=1}^{n} a_{kk}^{(k)}$$

$$(L^{(1)})^{-1} \qquad (L^{(1)})^{-1} L^{(1)} A^{(1)} = (L^{1})^{-1} A^{(2)}$$

$$= \begin{bmatrix} 1 & & & \\ \frac{a_{21}^{(1)}}{a_{11}^{(1)}} & 1 & 0 & \\ \vdots & & \ddots & \\ \frac{a_{n1}^{(1)}}{a_{11}^{(1)}} & 0 & & 1 \end{bmatrix} \qquad A^{(1)}$$

Unit lower triangular matrix

$$L^{(n-1)} \cdots L^{(1)} A = A^{(n)} = \begin{bmatrix} a_{11}^{(1)} & - - - & a_{1n}^{(1)} \\ & a_{22}^{(2)} \cdots & a_{2n}^{(2)} \\ 0 & & \\ & & a_{nn}^{(n)} \end{bmatrix} = U$$

$$A = \underline{\left(L^{(n-1)} \cdots L^{(1)}\right)^{-1}} A^{(n)}$$ = An upper triangular matrix

$$= \left(L^{(1)}\right)^{-1}\left(L^{(2)}\right)^{-1} \cdots \left(L^{(n-1)}\right)^{-1} U = LU$$

$$= \begin{bmatrix} 1 & & & \\ \frac{a_{21}^{(1)}}{a_{11}^{(1)}} & 1 & & \\ \frac{a_{i1}^{(1)}}{a_{11}^{(1)}} & & \ddots & \\ \vdots & & & \\ \frac{a_{n1}^{(1)}}{a_{11}^{(1)}} & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & \frac{a_{32}^{(2)}}{a_{22}^{(2)}} & 1 & 0 & \\ 0 & & & \ddots & \\ \vdots & & & & \\ 0 & \frac{a_{n2}^{(2)}}{a_{22}^{(2)}} & 0 & & 1 \end{bmatrix} \cdots$$

$$= \begin{bmatrix} 1 & & & \\ \frac{a_{21}^{(1)}}{a_{11}^{(1)}} & 1 & & 0 \\ \vdots & \vdots & 1 & \\ & & & \ddots \\ \vdots & \frac{a_{n2}^{(2)}}{a_{22}^{(2)}} & & \ddots \end{bmatrix}$$

$A = LU$

$\det(A) = \det(L)\det(U)$

$= \det(U)$

by product!

Just stack columns together.