Vector norms : A special function with (i) — (iii) properties.

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad p>0,$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| \ ?$$

Can you prove it?

Ex: $\|x\|_p$, $p = 1, 2, \infty$

$$\vec{x} = \begin{bmatrix} -5 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad \|x\|_1 = 8, \quad \|x\|_2 = \sqrt{30} = 5.\ldots, \quad \|x\|_\infty = 5$$

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1$$

$$f(x) \in L^2(a,b), \quad \|f\|_p = \left(\int_a^b |f|^p \, dx\right)^{1/p}, \quad p>0$$

1, 2, are called average norms

$f(x) = |x|$, non-differentiable at $x=0$

$\|x\|_2$ is differentiable, used the most

$\|E\|_\infty$ is the strongest norm in measuring errors. — Finite difference $\rightarrow \| \|_\infty$

Finite element methods $\rightarrow \underline{L^p}, \underline{p=2}$

$\rightarrow H^1(\Omega)$

In optimization, minimize $\| \|_1$

$\| \|_\infty$

sharp corners etc. Research topic

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \quad \text{in } \mathbb{R}^n$$

✓

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{x_1^2 + x_2^2 - \cdots x_n^2}$$

$$= \sqrt{x_1^2 + \cdots \underline{\underline{\|x\|_\infty^2}} + \cdots + x_n^2}$$

$$\ge \|x\|_\infty$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \le \sqrt{\sum_{i=1}^{n} \|x\|_\infty^2} = \sqrt{n}\, \|x\|_\infty$$

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\, \|x\|_\infty$$ in finite dimension

<u>Thm:</u> In a finite dimensional space, all vector norms are equivalent, that is, there are two positive constant. c, C such that

$$c\, \|x\|_{II} \le \|x\|_I \le C\, \|x\|_{II}$$

Hint of proof:

$$\frac{1}{C} \le \|x\|_I \le \|x\|_{II} \le \frac{1}{c}\, \|x\|_I \;✓$$

$\|x\|_2$ $\quad \|x\|_1 \le ? \ \|x\|_2$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \cdot 1$$

$$|\vec{x}| = \begin{bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{bmatrix}. \quad \vec{y} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Euclidian

$(a, b) \le \|a\|_2 \|b\|_2$

$$= (|\vec{x}|, \vec{y}) = |\vec{x}|^T y$$

$$\le \sqrt{\sum_{i=1}^{n} |x_i|} \cdot \sqrt{\sum_{i=1}^{n} 1}$$

$$= \sqrt{n} \ \|x\|_2$$

$$\|x\|_1 \le \sqrt{n} \|x\|_2 \qquad C = \sqrt{n}$$

Key: Cauchy-Schwartz

$(a+b+c)^2$

$= a^2 + b^2 + c^2$
$+ ab + ac + bc$
$+$

$$\|x\|_1 \ge \|x\|_2$$

$$\|x\|_1^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2 = \sum_{i=1}^{n} |x_i|^2 + 2 \cancel{\sum_{i<j}^{n} |x_i|} |x_j|$$

$$\ge \|x\|_2^2 \qquad \ge 0$$

$$\frac{1}{\sqrt{n}} \|x\|_1 \le \|x\|_2 \le \|x\|_1$$

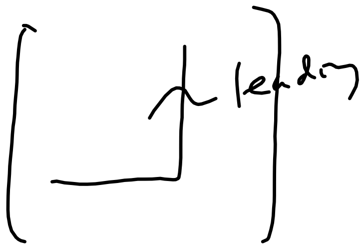
$$\|x\|_2 \le \|x\|_1 \le \sqrt{n} \|x\|_2$$

Matrix Norms $\quad A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad A = \{a_{ij}\} = [\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n]$$

<u>Method 1:</u> Treat as a big vector

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix},$$




$$\|A\|_F = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{m} (a_{ij})^2} = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

$\|x\|_2$

If $m = n$.

$\|I\|_F = \sqrt{n} \neq 1.$ $\quad Ax, \quad \|Ax\|$

$\|A\|, \|x\|$

A matrix norm is a special function of a matrix $A$, or its components $\{a_{ij}\}$

1. $f(A) \geq 0$, $f(A) = 0$ iff $A = 0$,
2. $f(\alpha A) = |\alpha| f(A)$, for any $\alpha \in R$
3. $f(A+B) \leq f(A) + f(B)$ $\quad \forall \alpha \in R$

$\rightarrow$ Then we write $f(A) = \|A\|$

Thm: Given a vector norm in $R^m$, the matrix function $f(A)$, $A \in R^{m \times n}$, $\underline{Ax} \in R^m$

$$f(A) = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \qquad m=n$$

is a matrix-norm. ✓

Comments/Hints for HW1, Due Wednesday

Matrix norms, $A \in \mathbb{R}^{n \times n}$, associated matrix-norm. Given a vector norm $\|x\|$ in $\mathbb{R}^n$, the matrix norm

$$f(A) = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|}{\underline{\underline{\|x\|}}} \qquad \sup_{x \in (0,1)} \frac{1}{x} = \infty$$

supremum

$$\sup_{x \in (0,2)} \cos \frac{1}{x} = 1$$

$$= \sup \left\| A \frac{x}{\underline{\underline{\|x\|}}} \right\| \qquad \|\alpha x\| = |\alpha| \, \|x\|$$

$$= \sup_{\|y\|=1} \|Ay\| = \max_{\|y\|=1} \|Ay\|$$

$\downarrow ?$

$$= \max_{\|x\|=1} \|Ax\|$$

(1) $f(A) \geq 0$ obvious! $f(A)=0$ iff $A=0$.

$f(A)=0$. $\sup\limits_{x\neq 0} \frac{\|Ax\|}{\|x\|}=0$. $\Rightarrow A=0$

$x=1$

If not true, then there is $a_{ij}\neq 0$, $\underline{Ax}$ $(a_{ij})$

Take $x=e_j=\begin{bmatrix}0\\0\\ \vdots\\1\\0\\ \vdots\\0\end{bmatrix} j$, $Ax=\begin{pmatrix}a_{1j}\\a_{2j}\\ \vdots \\ a_{ij}\\ \vdots\\ a_{nj}\end{pmatrix}\neq 0$

$\|Ax\|\neq 0$, $\|x\|\neq 0$, $\frac{\|Ax\|}{\|x\|}\neq 0 \Rightarrow \sup\limits_{x\neq 0}\frac{\|Ax\|}{\|x\|}\neq 0$

Contridiction!

(ii) $f(\alpha A)=\sup\limits_{x\neq 0}\frac{\|\alpha Ax\|}{\|x\|} = \sup\limits_{\substack{x\in R^n\\ x\neq 0}} |\alpha| \frac{\|Ax\|}{\|x\|}$

$=|\alpha|\sup\limits_{x\neq 0}\frac{\|Ax\|}{\|x\|}=|\alpha| f(A)$

(iii) $f(A+B) \leq f(A)+f(B)$

$$\sup_{x \neq 0} \frac{\|(A+B)x\|}{\|x\|} = \sup_{x \neq 0} \frac{\|Ax + Bx\|}{\|x\|}$$

$$\leq \sup_{x \neq 0} \left( \frac{\|Ax\| + \|Bx\|}{\|x\|} \right) \leq \sup \frac{\|Ax\|}{\|x\|} + \sup \frac{\|Bx\|}{\|x\|}$$

1, 2 2

3 1 $\frac{\cancel{3}}{5}$

4 3

4

$f(A)$ is a matrix norm!

$$\|I\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{x \neq 0} \frac{\|Ix\|}{\|x\|} = 1$$

(2) $\|Ax\| \leq \|A\| \, \|x\|$

A particular one

Proof

$$\underline{\|A\|} = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \geq \underline{\frac{\|Ax^*\|}{\|x^*\|}}$$

$$\|A\| \, \|x^*\| \geq \|Ax^*\|$$

$\|x\|_p \rightarrow \|A\|_p$

$p = 1, 2, \infty$

Thm: $\|A\|_\infty$, Add magnitude of rows, then pick up the largest.

rows $\rightarrow$ columns $\|A\|_1$

$$\|A\|_\infty = \max\left\{ \sum_{j=1}^{n} |a_{1j}|, \sum_{j=1}^{n} |a_{2j}|, \dots \sum_{j=1}^{n} |a_{nj}| \right\}$$

$$= \max_{1\le i\le n}\left\{ \sum_{j=1}^{n} |a_{ij}| \right\}$$

$$\|A\|_1 = \max\left\{ \sum_{i=1}^{n} |a_{i1}|, \sum_{i=1}^{n} |a_{i2}|, \dots \sum_{i=1}^{n} |a_{in}| \right\}$$

$$= \max_{1\le j\le n}\left\{ \sum_{i=1}^{n} |a_{ij}| \right.$$

Ex: 

$$A = \begin{pmatrix} -5 & 0 & 7 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} 12 \\ 5 \\ 1 \end{matrix}$$

$$\begin{matrix} 6 & 3 & 9 \end{matrix}$$

$\|A\|_\infty = 12$

$\|A\|_1 = 9$

Q. When $\|A\|_\infty = \|A\|_1$, $A = A^T$ in $\mathbb{R}$, $A = A^H$ in $\mathbb{C}$.

Prove one of them, $\|A\|_\infty$.

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty \overset{?}{=} \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

$\exists$ $i_0$ such that $M = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{i_0 j}|$

Step 1 $\max_{\|x\|_\infty = 1} \|Ax\|_\infty \le M$

Step 2 Find the particular $x^*$ such that

$\|Ax^*\|_\infty = M$. $\|x^*\|_\infty = 1$ $|x_j| \le 1$

$$\|Ax\|_\infty = \left\| \begin{bmatrix} \sum a_{1j} x_j \\ \sum a_{2j} x_j \\ \vdots \\ \sum a_{nj} x_j \end{bmatrix} \right\|_\infty = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

$$\le \max \sum |a_{ij}|$$

$$= M$$