Ex. $A = \begin{bmatrix} 1 & -1 & 6 \\ 2 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, Use Gaussian elimination with partial pivoting to find $PA = LU$, and $\det(LU)$

Solution.

$$\begin{bmatrix} 1 & -1 & 6 \\ 2 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -1 & 6 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{GE} \begin{bmatrix} 2 & 0 & 2 \\ \frac{1}{2} & -1 & 5 \\ \frac{1}{2} & 2 & 3 \end{bmatrix} \xrightarrow{P_{23}} \begin{bmatrix} 2 & 0 & 2 \\ \frac{1}{2} & 2 & 3 \\ \frac{1}{2} & -1 & 5 \end{bmatrix}$$

Entire rows

$$\xrightarrow{GE} \begin{bmatrix} 2 & 0 & 2 \\ \frac{1}{2} & 2 & 3 \\ \frac{1}{2} & -\frac{1}{2} & \frac{13}{2} \end{bmatrix}$$

$$P = P_{23} P_{12} = P_{23} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{13}{2} \end{bmatrix}$$

$$\det(A) = (-1)^2 \cdot 2 \cdot 2 \cdot \frac{13}{2} = 26$$

verify: $P_{23} P_{12} A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & \cancel{2} & 4 \\ 1 & -1 & 6 \end{bmatrix}$

Solve $Ax = b$, $\quad b = \begin{bmatrix} 6 \\ 4 \\ 7 \end{bmatrix}$

$Ax = b$. $\qquad P_{23} P_{12} A x = P_{23} P_{12} b$

$$\begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} \to \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$

$$L \underline{Ux} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$

$$L y = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} y = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix},$$

$y_1 = 4$

$y_2 = 7 - 2 = 5$

$y_3 = 6 - 2 + \frac{5}{2}$

$= \frac{13}{2}$

$$Ux = y$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{13}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ \frac{13}{2} \end{bmatrix}$$

$x_3 = 1$

$x_2 = (5 - 3)/2 = 1$

$x_1 = (4 - 2 \cdot 1)/2 = 1$