An Introduction to Partial Differential Equations

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January 8, 2018

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Contents

Preface

The purpose of this book is to provide an introduction to partial differential equations (PDE) for one or two semesters. The book is designed for undergraduate or beginning level of graduate students, and students from interdisciplinary areas including engineers, and others who need to use partial differential equations, Fourier series, Fourier and Laplace transforms. The prerequisite is a basic knowledge of calculus, linear algebra, and ordinary differential equations.

The text book aims to be practical, elementary, and reasonably rigorous; the book is concise that describes fundamental solution techniques for first order, second order, linear partial differential equations for general solutions, fundamental solutions, solution to Cauchy (initial value) problems, and boundary value problemsfor different PDEs one and two dimensions and different coordinates systems. The analytic solution to boundary value problems are based Sturm-Liouville eigenvalue problems and series solutions. The book is companied with enough well tested Maple files and some of Matlab codes that are available online. The use of Maple makes the complicated series solution simple, interactive, and visible. These features distinguish the book from other textbooks available in the related area.

While there are many PDE textbooks around, many of them cover either too much materials or too difficult. We propose to have a practical, elementary, and reasonably rigorous; the book is concise that describes fundamental solution techniques and combine with the use of Maple. We hope to have a 200-page book with independent Maple files.

This is a textbook based on materials that the authors have used in teaching undergraduate courses on partial differential equations at North Carolina State University.

A web-site http://www4.ncsu.edu/~zhilin/PDE_Book has been set up.

We would like to thank my students for proofreading the book.

Chapter 1

Introduction

A *differential equation* involves the ordinary derivatives of an unknown function of one independent variable (say $u(x)$), or the partial derivatives of an unknown function of more than one independent variable (say $u(x, y)$, or $u(t, x)$, or $u(t, x, y, z)$ etc.). Differential equations have been used extensively to model many problems in fluid and solid mechanics, biology, material sciences, economics, ecology, sports and computer sciences.[1] Examples include the Laplace equation for potentials, the Navier-Stokes equations in fluid dynamics, biharmonic equations for stresses in solid mechanics, and the Maxwell equations in electro-magnetics. For more examples and for mathematical theory of partial differential equations, we refer the reader to [**?**] and references therein.

However, although differential equations have such wide applications, too few can be solved exactly in terms of elementary functions such as polynomials, $\log x$, e^x, trigonometric functions ($\sin x$, $\cos x$, ...), etc. and their combinations. Even if a differential equation can be solved analytically, considerable effort and sound mathematical theory are often needed, and the closed form of the solution may even turn out to be too messy to be useful. If the analytic solution of the differential equation is unavailable or too difficult to obtain, or takes some complicated form that is unhelpful to use, we may try to find an approximate solution. There are two traditional approaches:

- Semi-analytic methods. Sometimes we can use series, integral equations, perturbation techniques, or asymptotic methods to obtain an approximate solution expressed in terms of simpler functions.
- Numerical solutions. Discrete numerical values may represent the solution to a certain accuracy. Nowadays, these number arrays (and associated tables or plots) are obtained using computers, to provide the effective solution of many

[1] There are other models in practice, for example statistical models.

problems that were impossible to obtain before.

In this book, we mainly adopt the first approach and focus on either analytic solutions or series solutions.

Some examples of ODE/PDE are as follows.

1. Initial value problems (IVP). The canonical first order system is

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \; ; \tag{1.1}$$

and a single higher order differential equation may be rewritten as a first order system. For example, a second order ordinary differential equation

$$\begin{aligned} &u''(t) + a(t)u'(t) + b(t)u(t) = f(t), \\ &u(0) = u_0, \quad \boxed{u'(0) = v_0}. \end{aligned} \tag{1.2}$$

is converted into a first order system by setting $y_1(t) = u$ and $y_2(t) = u'(t)$.

2. Boundary value problems (BVP). An example of an ODE BVP is

$$\begin{aligned} &u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \\ &u(0) = u_0, \quad u(1) = u_1; \end{aligned} \tag{1.3}$$

and a PDE BVP example is

$$\begin{aligned} &u_{xx} + u_{yy} = f(x, y), \quad (x, y) \in \Omega \\ &u(x, y) = u_0(x, y), \quad (x, y) \in \partial\Omega, \end{aligned} \tag{1.4}$$

in a domain Ω with boundary $\partial\Omega$. The above PDE is linear and classified as *elliptic*, and there are two other classifications for linear PDE, namely, *parabolic* and *hyperbolic*, as briefly discussed below. The PDE is called a 2D Poisson equation. If $f(x, y) = 0$, it is a 2D Laplace equation.

3. Boundary and initial value problems, *e.g.*,

$$\begin{aligned} &u_t = c^2 u_{xx} + f(x, t) \\ &u(0, t) = g_1(t), \quad u(1, t) = g_2(t), \qquad \text{BC} \\ &u(x, 0) = u_0(x), \qquad \text{IC}. \end{aligned} \tag{1.5}$$

We call $f(x, t)$ a source term. If $f(x, t) = 0$, the PDE is called a 1D heat equation. It is a parabolic PDE. Note that the PDE $u_t = -c^2 u_{xx}$ is called a backward heat equation. A nonzero perturbation at some time instances will result an exponential grow in the solution as t increases. A two dimensional heat equation has the following form

$$u_t = c^2\left(u_{xx} + u_{yy}\right). \tag{1.6}$$

4. Eigenvalue problems, *e.g.*,

$$\begin{aligned} &u''(x) = \lambda u(x), \\ &u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{1.7}$$

   In this example, both the function $u(x)$ (the *eigenfunction*) and the scalar λ (the *eigenvalue*) are unknowns.

5. Diffusion and reaction equations, *e.g.*,

$$\frac{\partial u}{\partial t} = \nabla \cdot (\beta \nabla u) + \mathbf{a} \cdot \nabla u + f(u) \tag{1.8}$$

   where $\mathbf{a}$ is a constant vector, $\nabla \cdot (\beta \nabla u)$ is a diffusion term, $\mathbf{a} \cdot \nabla u$ is called an advection term, and $f(u)$ a reaction term.

6. Wave equations in 1D has the following form

$$u_{tt} = c^2 u_{xx}. \tag{1.9}$$

   where c is called the wave speed. It is a hyperbolic ODE. A 2D wave equation is

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right). \tag{1.10}$$

7. Systems of PDE. The incompressible Navier-Stokes model is an important nonlinear example:

$$\begin{aligned} &\rho\left(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = \nabla p + \mu \Delta \mathbf{u} + \mathbf{F}, \\ &\nabla \cdot \mathbf{u} = 0. \end{aligned} \tag{1.11}$$

In this book, we will consider *linear* PDE in either one dimension (1D) or two dimensions (2D). A 2D linear PDE has the general form

$$\begin{aligned} &a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} \\ &\qquad + d(x,y)u_x + e(x,y)u_y + g(x,y)u(x,y) = f(x,y) \end{aligned} \tag{1.12}$$

where the coefficients are independent of $u(x,y)$ so the equation is linear in u and its partial derivatives. In the example above, the solution of the 2D linear PDE is sought in some bounded domain Ω, and the classification of the PDE form (1.12) is:

- Elliptic if $b^2 - ac < 0$ for all $(x, y) \in \Omega$,
- Parabolic if $b^2 - ac = 0$ for all $(x, y) \in \Omega$, and
- Hyperbolic if $b^2 - ac > 0$ for all $(x, y) \in \Omega$.

The appropriate solution method typically depends on the equation class. For the first order system

$$\frac{\partial \mathbf{u}}{\partial t} = A(\mathbf{x}) \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \tag{1.13}$$

the classification is determined from the eigenvalues of the coefficient matrix $A(\mathbf{x})$.

Chapter 2

First order PDEs

The simplest PDE may be the advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad \text{or} \quad u_t + au_x = 0, \tag{2.1}$$

where t and x are independent variables, $u(x,t)$ is the dependent variable that to be solved. In application, t often stands for the time, and x stands for the space, and a is called the wave speed. The PDE is called a one-dimensional, first order, linear, constant coefficient, and homogeneous PDE. Although there are two independent variables, it is called one-dimensional (1D) advection equation since the space It is a hyperbolic PDE. It is called an advection equations; or one-way wave equation, or a transport equation.

2.1 Method of changing variables

There are several ways to solve the PDE. One of them is the method of changing variables. The idea is to change the PDE to an ODE so that we can use the ODE solution method to solve it. A simplest way of changing variables is the following

$$\begin{aligned} &\xi = x - at, \\ &\eta = t, \end{aligned} \qquad \text{or} \qquad \begin{aligned} &x = \xi + a\eta, \\ &t = \eta \end{aligned} \tag{2.2}$$

Under such a transform, we have $u(x,t) = u(\xi + a\eta, \eta)$. We denote $U(\xi,\eta) = u(\xi + a\eta, \eta)$. Then using the chain rule, we can get

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = -a\frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}, \\ \frac{\partial u}{\partial x} &= \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}. \end{aligned}$$

Plug them into the original PDE (2.1), we would get

$$\frac{\partial U}{\partial \eta} = 0. \tag{2.3}$$

Integrate both sides above with respect to η, we get $U(\xi, \eta) = C$. Note that in an ODE, C is an arbitrary constant. But in PDE, it can be arbitrary differential function of ξ, denoted as $f(\xi)$! Thus we get the solution

$$u(x,t) = u(\xi + a\eta, \eta) = U(\xi, \eta) = f(\xi) = f(x - at). \tag{2.4}$$

It is straightforward to check that $u(x, y)$ above is indeed a solution to the PDE (2.1). It is called the *general solution* of the PDE since there is no condition attached to the problem.

Note that there are more than one ways of changing variables. In general, we can use

$$\xi = a_{11}x + a_{12}t \tag{2.5}$$

$$\xi = a_{21}x + a_{22}t, \tag{2.6}$$

where a_{ij} are parameters and $det(A) \neq 0$ for $A = \{a_{ij}\}$. We can choose a_{ij} so that the PDE in the new variables is simple, like an ODE, so that we can solve it easily. In the discussion above we have $a_{11} = 1$, $a_{12} = -a$, $a_{21} = 0$, and $a_{22} = 1$.

2.2 Solution to the Cauchy problems

A Cauchy problem is an initial value problem that is defined in the entire space, that is

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad -\infty < x < \infty \tag{2.7}$$

$$u(x,0) = u_0(x), \tag{2.8}$$

where $u_0(x)$ is a differentiable function in $(-\infty, \infty)$. Since we know the general solution is $u(x,t) = f(x - at)$, we have $u(x, 0) = f(x) = u_0(x)$. Thus the solution to the Cauchy problem is

$$u(x,t) = u_0(x - at), \tag{2.9}$$

where $u_0(x)$ is the initial condition. Which means that the solution at (x, t) is the same as the initial solution at $(x - at, 0)$. When $a > 0$, $x - at < x$, the solution propagates towards right without changing the shape. That is why it is called a one-way wave equation, or advection equation.

Example 2.1. *Let $a = 2$ and*

$$u_0(x) = \begin{cases} \sin x & -\pi \le x \le \pi \\ 0 & \textit{otherwise} \end{cases} \tag{2.10}$$

The solution to the Cauchy problem is

$$u_0(x,t) = \begin{cases} \sin(x-2t) & -\pi \le x - 2t \le \pi \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

In Figure 2.1, we plot the solution at $t = 0$ and $t = 1$, we can see that the solution is simply shifted to the right.

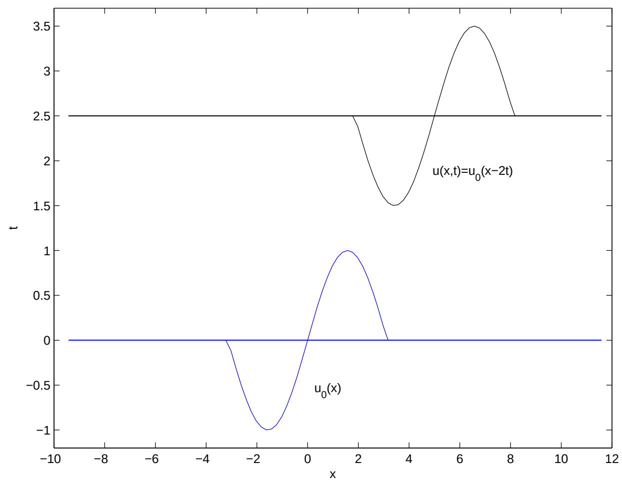

Figure 2.1. *Plot of the initial condition $u_0(x)$ and the solution $u(x,t)$ to the advection equation at $t = 2.5$ with the wave speed $a = 2$.*

2.3 Method of characteristic for advection equations

A characteristic to a PDE is a set in which the solution to the PDE is a constant (does not change). For first order PDE of the form $u_t + p(x,t)u_x = f(x,t)$, a characteristic is often a continuous curve $(t(s), x(s)$ for a parameterizations of s. Let us examine the advection PDE $u_t + au_x = 0$ first. Since along the characteristic, the solution $u(x,t) = C$, a constant. Differentiating it with respect to t we get

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0.$$

Since $u(x,t)$ is the solution to the PDE, we have to have $\frac{dx}{dt} = a$ or $x = at+\bar{C}$. Thus we have $\bar{C} = x - at$. Since $u(x,t)$ is a constant along the line (the characteristic), we have

$$u(x,t) = u(\bar{C},0) = u_0(\bar{C}) = u_0(x-at), \tag{2.12}$$

where $u_0(x)$ is the initial condition. Often we can simply write $\bar{C} = C$. Note that in PDE, an arbitrary constant often corresponding to an arbitrary function $C = f(x-at) = u(x,t)$. Once again, we get the general solution using a different method. It is important to know that along a curve $\bar{x} - x = a(\bar{t} - t)$, the solution $u(x,t)$ is a constant, which is the basis to determine appropriate boundary conditions for boundary value problems.

2.4 Solution of advection equation of boundary value problems.

Now consider the boundary value problem of an advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad 0 < x < L \tag{2.13}$$

$$u(x,0) = u_0(x), \qquad 0 < x < L, \tag{2.14}$$

for a positive constant L. We need one or two boundary conditions to make the problem well-posed, that is, the solution exists and it is unique. Given a point (x,t), $0 < x < L$ and $t > 0$, we can use the method of characteristic to track back the solution to either the initial condition or boundary condition whichever is the first hit by the characteristic in the domain.

For example, assume that $a > 0$, see the left diagram in Figure 2.2 for an illustration. The line $x = at$ passes through the origin and divide the domain in the first quadrant as two parts; one region we have $0 < at < x$; the other region is $0 < x < at$. In the first region, the line equation $x = at + C$ if we trace back from a point (x,t) will hit the x axis $(t = 0)$ fist at $x = C$ when $t = 0$. Thus the solution in this region is

$$u(x,t) = u(C,0) = u_0(C) = u_0(x-at), \quad \text{if } L > x > at > 0. \tag{2.15}$$

In the second region in the first quadrant $0 < x < at$, the line equation $x = at + C$ if we trace back from a point (x,t) will hit the t axis $(x = 0)$ fist at $x = 0$ when $t = -C/a$. Thus the solution in this region is

$$u(x,t) = u\left(0, -\frac{C}{a}\right) = g\left(-\frac{C}{a}\right) = g\left(t - \frac{x}{a}\right) \quad \text{if } 0 < x < at. \tag{2.16}$$

In summary, for $a > 0$, we need to prescribed a boundary condition at $x = 0$, say, $u(0,t) = g(t)$, then the solution is

$$u_0(x,t) = \begin{cases} u_0(x - at) & 0 < at < x < L,\ t > 0 \\ g\left(t - \dfrac{x}{a}\right) & 0 < x < \min\{at, L\} \text{ and } t > 0. \end{cases} \tag{2.17}$$

2.5 Boundary value problems of advection equation with $a < 0$. *

With similar discussions, the boundary value problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad 0 < x < L \tag{2.18}$$

$$u(x,0) = u_0(x), \qquad 0 < x < L, \tag{2.19}$$

with $a < 0$ requires a boundary condition at $x = L$, say, $u(L,t) = g_r(t)$ is given, see the right diagram in Figure 2.2 for an illustration.

The line equation $x = at + L$ passes through $(L, 0)$ and divides the domain in the first quadrant as two parts; one region we have $0 < x < at + L$; the other region is $0L + at < x < L$. In the first region, the line equation $x = at + C$ if we trace back from a point (x,t) will hit the x-axis $(t = 0)$ fist at $x = C$ when $t = 0$. Thus the solution in this region once again is

$$u(x,t) = u(C,0) = u_0(C) = u_0(x - at), \quad \text{if } 0 < x < at + L. \tag{2.20}$$

In the second region in the first quadrant $0 < x < at$, the line equation $x = at+C$ if we trace back from a point (x,t) will hit the $x = L$ axis $(x = L)$ fist at $t = (L-C)/a$. Thus the solution in this region is

$$u(x,t) = u\left(L, \frac{L-C}{a}\right) = g_r\left(\frac{L + (x - at)}{a}\right) = g\left(t - \frac{L-x}{a}\right) \quad \text{if } L + at < x < L.$$

The solution then is

$$u_0(x,t) = \begin{cases} u_0(x - at) & 0 < x < L + at;\ t > 0, \\ g_r\left(t + \dfrac{L-x}{a}\right) & \max\{0,\ L + at\} < x < L,\ t > 0. \end{cases} \tag{2.21}$$

2.6 Method of characteristic for general linear first order PDEs

Consider a general linear first order PDE

$$\frac{\partial u}{\partial t} + p(x,t)\frac{\partial u}{\partial x} = 0. \tag{2.22}$$

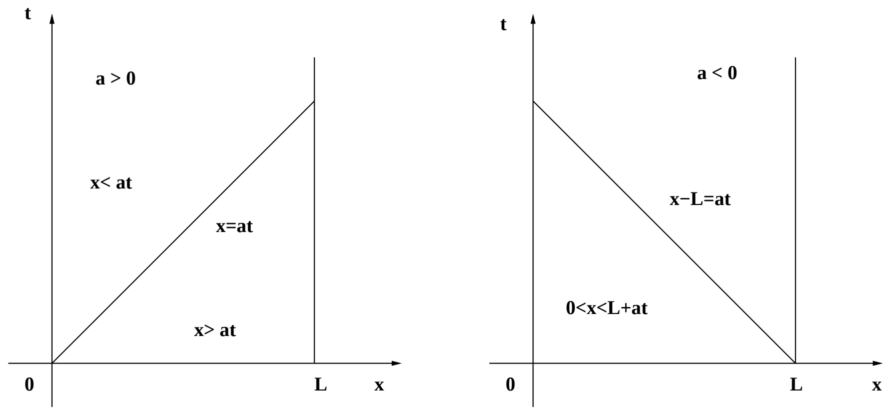

Figure 2.2. *Diagram of the two regions where the solution of the advection is determined either by the initial or a appropriate boundary condition. The left diagram is for $a > 0$ while the right is for $a < 0$.*

In the method of characteristic, we set $\frac{dx}{dt} = p(x,t)$. If we can solve this ODE to get $x - g(t) = C$. Then the general solution to the original problem is $u(x,t) = f(x - g(t))$ for any differentiable function $f(x)$.

Proof: If $u(x,t) = f(x - g(t))$ and $\frac{dx}{dt} = -g'(t) = p(x,t)$, then we have $\frac{\partial u}{\partial t} = f'g'(t) = f'p(x,t)$ and $\frac{\partial u}{\partial x} = f'$. Thus we have $\frac{\partial u}{\partial t} + p(x,t)\frac{\partial u}{\partial x} = f'(-p) + pf' = 0$.

Example 2.2. *Find the general solution to*

$$\frac{\partial u}{\partial t} + x^2 \frac{\partial u}{\partial x} = 0.$$

Find also the solution to the Cauchy problem if $u(x,0) = \sin x$.

Solution: We set $\frac{dx}{dt} = p(x,t) = x^2$ or $\frac{dx}{x^2} = dt$. We get $-\frac{1}{x} = t + C$ or $C = t - \frac{1}{x}$. The general solution is $u(x,t) = f(t - \frac{1}{x})$.

Since we have $u(x,0) = f(-1/x) = \sin x$. Let $y = -1/x$ we get $f(y) = -\sin y$. The solution to the Cauchy problem is $u(x,t) = -\sin \frac{1}{t-1/x} = -\sin \frac{x}{tx-1}$.

Example 2.3. *Find the general solution to*

$$\frac{\partial u}{\partial t} + t^2 \frac{\partial u}{\partial x} = 0.$$

Find also the solution to the Cauchy problem if $u(x,0) = \sin x$.

Solution: We set $\frac{dx}{dt} = p(x,t) = t^2$ or $x = t^3/3 + C$. We get $C = x - t^3/3$. The general solution is $u(x,t) = f(x - \frac{t^3}{3})$.

Since we have $u(x,0) = f(x) = \sin x$. The solution to the Cauchy problem is $u(x,t) = \sin\left(x - \frac{t^3}{3}\right)$.

2.6.1 Solution to first order linear non-homogeneous PDEs with constant coefficients

Using the method of changing variables, we can transform a first order linear non-homogeneous PDEs with constant coefficients

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + bu = f(x,t) \tag{2.23}$$

to an ODE. Thus we can solve the ODE to get the general solution to the PDE. With the same way of changing variables

$$\begin{aligned} \xi &= x - at, \\ \eta &= t, \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \xi + a\eta, \\ t &= \eta. \end{aligned} \tag{2.24}$$

Under such a transform, we have $u(x,t) = u(\xi + a\eta, \eta)$. We denote $U(\xi,\eta) = u(\xi + a\eta, \eta)$. Then using the chain rule, we can get

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = -a\frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}, \\ \frac{\partial u}{\partial x} &= \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}. \end{aligned}$$

Plug them into the original PDE (2.23), we would get

$$\frac{\partial U}{\partial \eta} + bU = f(\xi + a\eta, \eta) = F(\xi, \eta). \tag{2.25}$$

The equation above is actually ordinary differential equation with respect to η (treating ξ as a constant. If we can solve the ODE above, we can get the general solution to the original PDE.

Example 2.4. *Find the general solution to*

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} - u = t.$$

Solution: With the changing variable $\xi = x - 2t$, $\eta = t$, the PDE becomes

$$\frac{\partial U}{\partial \eta} - U = \eta.$$

It is a non-homogeneous ODE and the solution can be expressed as

$$U = U_h + U_p$$

in which U_h is the homogeneous solution to $\frac{\partial U}{\partial \eta} - U = 0$ and U_p is a particularly solution to the ODE. It is easy to get $U_h(\xi, \eta) = g(\xi)e^{\eta}$. From the ODE technique, we can set

$$U_p = A\eta + B$$

for two constants A and B. Plug this into the ODE and matching terms on both sides, we get $A = -1$, $B = -1$. Thus the solution in the new variables is

$$U_h(\xi, \eta) = g(\xi)e^{\eta} - \eta - 1.$$

Thus the general solution to the PDE then is

$$u(x, t) = g(x - 2t)e^{t} - t - 1.$$

Example 2.5. *Find the general solution to*

$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial u}{\partial x} + 4u = xt.$$

Solution: With the changing variable $\xi = x + \frac{1}{2}t$, $\eta = t$, the PDE becomes

$$\frac{\partial U}{\partial \eta} + 4U = \left(\xi - \frac{1}{2}\eta\right)\eta.$$

It is a non-homogeneous ODE and the solution can be expressed as

$$U = U_h + U_p$$

in which U_h is the homogeneous solution to $\frac{\partial U}{\partial \eta} + 4U = 0$ and U_p is a particularly solution to the ODE. It is easy to get $U_h(\xi, \eta) = g(\xi)e^{-4\eta}$. From the ODE technique, we can set

$$U_p = A\eta^2 + B\eta + C$$

where A, B, and C are constants. Plug this into the ODE and matching terms on both sides, we get $A = -1/8$, $B = \frac{\xi}{4} + \frac{1}{16}$, $C = -\frac{\xi}{16} - \frac{1}{64}$. Thus the solution in the new variables is

$$U_h(\xi, \eta) = g(\xi)e^{-4\eta} - \frac{\eta^2}{2} + \left(\frac{\xi}{4} + \frac{1}{16}\right)\eta - \frac{\xi}{16} - \frac{1}{64}.$$

Thus the general solution to the PDE then is

$$u(x, t) = g\left(x + \frac{t}{2}\right)e^{-4t} - \frac{t^2}{2} + \left(\frac{x + t/2}{4} + \frac{1}{16}\right)t - \frac{x + t/2}{16} - \frac{1}{64}.$$

Chapter 3

Solution to 1D Wave equations

A one-dimensional wave equation has the following form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

where c is called the wave number. The PDE is a second order, linear, constant, homogeneous one. According to the criteria, the PDE is a hyperbolic one. We first find the general solution for which no constraints are imposed.

We can use the method of changing variables to simplify the PDE by setting

$$\begin{aligned} \xi &= x - ct, \\ \eta &= x + ct, \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= \frac{\xi + \eta}{2}, \\ t &= \frac{\eta - \xi}{2c}. \end{aligned} \tag{3.2}$$

Under such a transform, we have $u(x,t) = u\left(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2c}\right) = U(\xi, \eta)$. Then using the chain rule, we can get

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial U}{\partial \xi} + c\frac{\partial U}{\partial \eta}, \\ \frac{\partial u}{\partial x} &= \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}. \end{aligned}$$

Differentiating again, we get

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= (-c)\frac{\partial^2 U}{\partial \xi^2}\frac{\partial \xi}{\partial t} + (-c)\frac{\partial^2 U}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial t} + c\frac{\partial^2 U}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial t} + c\frac{\partial^2 U}{\partial \eta^2}\frac{\partial \eta}{\partial t} \\ &= c^2\frac{\partial^2 U}{\partial \xi^2} - 2c^2\frac{\partial^2 U}{\partial \xi \partial \eta} + c^2\frac{\partial^2 U}{\partial \eta^2}, \end{aligned}$$

assuming both $\frac{\partial^2 U}{\partial\xi\partial\eta}$ and $\frac{\partial^2 U}{\partial\eta\partial\xi}$ are continuous so that they are the same. Similarly we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 U}{\partial \xi^2} + 2\frac{\partial^2 U}{\partial\xi\partial\eta} + \frac{\partial^2 U}{\partial \eta^2}$$

Plug them into the original PDE, we get

$$4c^2 \frac{\partial^2 U}{\partial\xi\partial\eta^2} = 0, \qquad \text{or} \qquad \frac{\partial^2 U}{\partial\xi\partial\eta} = 0$$

since $c \neq 0$. We integrate with respect to η we get $\frac{\partial U}{\partial\xi\partial} = f(\xi)$; and integrate it with respect to ξ we get

$$U(\xi,\eta) = \int f(\xi)d\xi + G(\eta) = F(\xi) + G(\eta)$$

since $\int f(\xi)d\xi$ is still a function of ξ. Finally, we get back to the original variables to get the general solution

$$u(x,t) = F(x-ct) + G(x+ct) \tag{3.3}$$

for any twice one dimensional differentiable functions $F(x)$ and $G(x)$.

3.1 Solution to the wave equations: the Cauchy problems

A Cauchy problem (an initial value problem) of the 1D wave equation has the following form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty \tag{3.4}$$

$$u(x,0) = f(x); \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad -\infty < x < \infty \tag{3.5}$$

where $f(x)$ and $g(x)$ are given initial conditions. The solution to the Cauchy problem can be represented by the D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2c}\int_{x-at}^{x+at} g(s)ds. \tag{3.6}$$

Proof: First we check the boundary condition. We have

$$u(x,0) = \frac{1}{2}\left(f(x) + f(x)\right) + 0 = f(x)$$

since the integration is zero if the lower and upper limit of the integration are the same. Secondly, we differentiate the equality above with respect to t to get

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\left(f'(x)(-c) + f'(x)c\right) + \frac{1}{2c}\left(cg(x) - g(x)(-c)\right) = g(x).$$

To prove the D'Alembert's formula satisfies the wave equation, we just need to solve for $F(x-ct)+G(x+ct)$ of the general solution in terms of $f(x)$ and $g(x)$. From the initial condition, we already have

$$u(x,0) = F(x) + G(x) = f(x). \tag{3.7}$$

Differentiating the general solution with respect to t, we get

$$\frac{\partial u}{\partial t} = F'(x-ct)(-c) + G'(x+ct)c. \tag{3.8}$$

At $t = 0$, we get

$$\frac{\partial u}{\partial t}(x,0) = -F'(x)c + G'(x)c = g(x), \text{ or } F(x) - G(x) = -\frac{1}{c}\int_0^x g(s)ds. \tag{3.9}$$

Along with $F(x) + G(x) = f(x)$ which is $F'(x) + G'(x) = f'(x)$ we solve for $F(x)$ and $G(x)$. Add the two identities together we get

$$2G'(x)c = f'(x)c + g(x) \quad \text{or} \quad G'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x)$$

From this we get

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds + A$$

where A is a constant. Since we have

$$F(x) = f(x) - G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds - A.$$

Plug them into the general solution, we get

$$\begin{aligned}
u(x,t) = F(x-ct) + G(x+ct) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)ds - A \\
&\quad + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + A \\
&= \frac{1}{2}\left(f(x+ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^0 g(s)ds + \frac{1}{2c}\int_0^{x+ct} g(s)ds \\
&= \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2c}\int_{x-at}^{x+at} g(s)ds.
\end{aligned}$$

Example 3.1. *Solve the Cauchy problem for the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty$$

$$u(x,0) = \begin{cases} \sin x & \text{if } |x| \le \pi \\ 0 & \text{otherwise;} \end{cases} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Solution: The solution is simply

$$u(x,t) = \frac{1}{2}\left(f(x-2t) + f(x+2t)\right)$$

. The If t is large enough, then the non-zero regions of $f(x-2t)$ and $f(x+2t)$ do not overlap. We see clearly a single sine wave in the domain $(-\pi,\pi)$ propagates to the right and left with half the magnitude, see Fig. 3.1. We call $f(x-ct)$ the right-going wave, while $f(x+2t)$ the left-going wave.

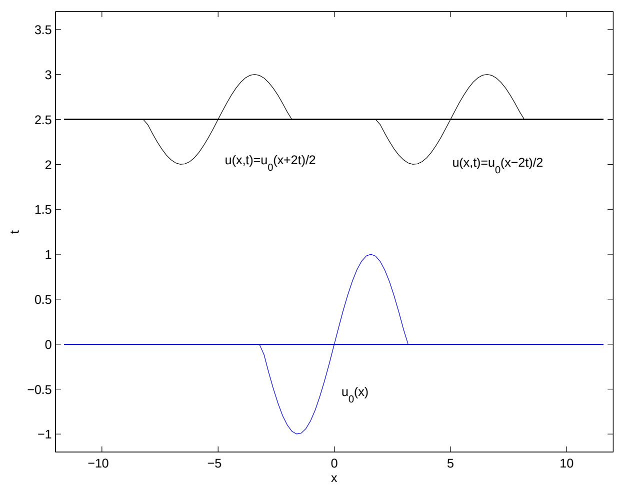

Figure 3.1. *Plot of the initial condition* $u_0(x)$ *and the* $u(x,t)$ *to the 1D wave solution at* $t = 2.5$ *with the wave speed* $c = 2$.

Example 3.2. *Solve the Cauchy problem for the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty$$

$$u(x,0) = \sin x \qquad \frac{\partial u}{\partial t}(x,0) = xe^{-x^2}.$$

Solution: The solution is

$$u(x,t) = \frac{1}{2}\left(\sin(x-\sqrt{2}t)+\sin(x+\sqrt{2}t)\right) + \frac{1}{2\sqrt{2}}\int_{x-\sqrt{2}t}^{x+\sqrt{2}t} se^{-s^2}ds$$
$$= \frac{1}{2}\left(\sin(x-\sqrt{2}t)+\sin(x+\sqrt{2}t)\right) + \frac{1}{4\sqrt{2}}\left(e^{-(x-\sqrt{2}t)^2} - e^{-(x+\sqrt{2}t)^2}\right).$$

Example 3.3. *Solve the Cauchy problem for the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty$$

$$u(x,0) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

In Figure 3.2, we show the plot of the solution at time $t = 0$, $t = 0.3$, and $t = 5$. We can see clearly how the one wave split into two with half strength towards left $(x - t))$ and right $(x - t))$. A Matlab movie file is also available (wave_piece.m and fp.m).

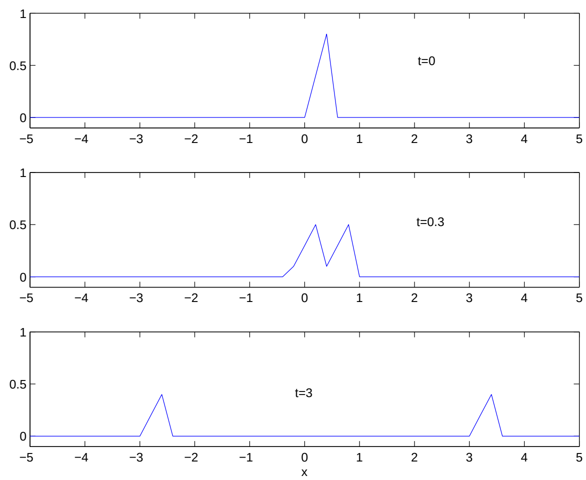

Figure 3.2. *Plot of the wave propagation at time $t = 0$, $t = 0.3$, and $t = 5$. We can see clearly how the one wave split into two with half strength towards left $(x - t))$ and right $(x - t))$.*

3.2 Normal Modes solution: Solution to 1D wave equations with special initial conditions

Now consider the boundary value problem of an advection equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L \\
&u(0,t) = 0, \qquad u(L,t) = 0. \\
&u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L,
\end{aligned} \tag{3.10}$$

for a positive constant L. An application is an elastic string of a length L with two ends fixed, which corresponds to the homogeneous boundary conditions.

Consider a special function

$$u_n(x,t) = \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}, \tag{3.11}$$

for a non-zero integer n. It is obviously that $u_n(0,t) = u(L,t) = 0$ and $u_n(x,t)$ satisfies the PDE (3.10). Note that $u_n(x,0) = \sin \frac{n\pi x}{L}$ and $\frac{\partial u}{\partial t}(x,0) = 0$. Thus, if $f(x) = \sin \frac{n\pi x}{L}$ and $g(x) = 0$, then $u_n(x,t)$ is the solution to the initial-boundary value problem (3.10). Such a solution is called the normal modes of the initial-boundary value problem.

Similarly,

$$\bar{u}_n(x,t) = \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L} \tag{3.12}$$

also satisfies the boundary condition and the PDE. Now we have $\bar{u}_n(x,0) = 0$ and $\frac{\partial \bar{u}}{\partial t}(x,0) = \frac{nc\pi}{L} \sin \frac{n\pi x}{L}$. Thus, if $f(x) = 0$ and $g(x) = \frac{nc\pi}{L} \sin \frac{n\pi x}{L}$, then $\bar{u}_n(x,t)$ is the solution to the initial-boundary value problem (3.10).

Example 3.4. *If $f(x) = \frac{1}{2} \sin \frac{5\pi x}{L}$ and $g(x) = 0$, find the solution to the initial-boundary value problem (3.10).*

Solution: *The solution is*

$$u(x,t) = \frac{1}{2} \sin \frac{5\pi x}{L} \cos \frac{5\pi ct}{L}.$$

Example 3.5. *If $f(x) = 0$ and $g(x) = \frac{1}{2} \sin \frac{5\pi x}{L}$, find the solution to the initial-boundary value problem (3.10).*

Solution: *The solution is*

$$u(x,t) = \frac{L}{10\pi c} \sin \frac{5\pi x}{L} \sin \frac{5\pi ct}{L}.$$

Example 3.6. *If* $f(x) = \sin\frac{5\pi x}{L} - 10\sin\frac{20\pi x}{L}$ *and* $g(x) = \frac{1}{2}\sin\frac{15\pi x}{L}$, *find the solution to the initial-boundary value problem (3.10).*

Solution: *The solution is*

$$u(x,t) = \sin\frac{5\pi x}{L}\cos\frac{5\pi ct}{L} - 10\sin\frac{20\pi x}{L}\cos\frac{20\pi ct}{L} + \frac{L}{30\pi c}\sin\frac{15\pi x}{L}\sin\frac{15\pi ct}{L}.$$

This is because the PDE is linear, homogeneous, and with homogeneous boundary conditions.

Challenge: How about the normal modes of

$$\tilde{u}_n(x,t) = \cos\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}, \qquad \hat{u}_n(x,t) = \cos\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}? \tag{3.13}$$

From the superposition, we know that the linear combination

$$u_N(x,t) = \sum_{n=1}^{N}\left(a_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\right) \tag{3.14}$$

is the solution to the initial-boundary value problem (3.10) with special initial condition

$$u_N(x,0) = f(x) = \sum_{n=1}^{N} a_n\sin\frac{n\pi x}{L}$$

$$\frac{\partial u_N}{\partial t}(x,0) = g(x) = \sum_{n=1}^{N} b_n\frac{L}{n\pi c}\sin\frac{n\pi x}{L}$$

What should we do for other general $f(x)$ and $g(x)$? We can use the separation variable and Fourier expansion ($N \to \infty$).

Chapter 4

Orthogonal functions and expansions, and Sturm-Liouville problems

For 1D wave equations with homogeneous boundary conditions, if $f(x) = e^x$ or even $f(x) = \sum_{i=0}^{N} a_i x^i$, we can not using the combination of normal modes solutions unless we have infinite terms of them. Nevertheless, we can get a series solution if we can have

$$e^x = \sum_{n=0}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{4.1}$$

This is called an orthogonal functions expansion of e^x. How do we get those orthogonal functions? The answer is from the Sturm-Liouville eigenvalue problems.

Also consider the method of separation of variables for the 1D wave equation. We try the solution of the form $u(x,T) = T(t)X(x)$. To satisfy the boundary conditional, we should have $X(0) = 0$ and $X(L) = 0$. Thus we have $\frac{\partial^2 u}{\partial t^2} = T''(t)X(x)$ and $\frac{\partial^2 u}{\partial x^2} = T(t)X''(x)$. Plug them into the wave equation we get

$$T''(t)X(x) = c^2 T(t)X''(x), \qquad \text{or} \qquad \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

This is possible only if

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

for some constant λ. Thus we have

$$X''(x) - \lambda X(x) = 0, \qquad X(0) = X(L) = 0. \tag{4.2}$$

The solution is

$$X(x) = C_1 e^{\sqrt{\lambda}} + C_2 e^{-\sqrt{\lambda}}$$

if $\lambda \geq 0$ which implies $X(x) = 0$ due to the boundary condition. $X(x) = 0$ is a trivial solution. If $\lambda < 00$, then the solution is

$$X(x) = C_1 \cos \sqrt{-\lambda} x + C_2 \sin \sqrt{-\lambda} x.$$

$X(0) = 0$ implies that $C_1 = 0$ and $X(x) = C_2 \sin \sqrt{-\lambda} x$. $X(L) = 0$ implies that $X(x) = \sin \sqrt{-\lambda} L =$. This is a special Sturm-Liouville eigenvalue problem. We get

$$\sqrt{-\lambda L} = n\pi, \quad \text{or} \quad -\lambda = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \cdots,$$

$$u_n(x) = \sin \frac{n\pi x}{L}$$

which satisfies the ODE and the boundary conditions. the set $\left\{\sin \frac{n\pi x}{L}\right\}$ are called the eigenfunctions.

4.1 Orthogonal functions

Orthogonal functions are similar to orthogonal basis in R^n space in linear algebra. Examples and applications include Fourier series. One of notable application is that we can expand functions in terms of orthogonal functions. Orthogonal functions are also intensively applied in computational mathematics as approximation tools.

In R^n space which is all the column vectors with n components. The simplest orthogonal basis are $\{\mathbf{e}_i\}$. For example, if $n = 3$. we have

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

We have

$$(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i^T \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

. For any vector $\mathbf{a} = [a_1, a_2, a_3]^T$, we have $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$. If $\mathbf{b} = \{b_i\}$ is a vector, then $(a, b) = \sum_{i=1}^3 a_i b_i$.

There are other orthogonal basis in R^3, for example,

$$\tilde{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix},$$

also form a unit-orthogonal basis. How do we express any vector in terms of $\{\tilde{\mathbf{e}}_i\}$?

$$\mathbf{a} = \alpha_1\tilde{\mathbf{e}}_1 + \alpha_2\tilde{\mathbf{e}}_2 + \alpha_3\tilde{\mathbf{e}}_3, \qquad \alpha_i = \frac{(\mathbf{a}, \tilde{\mathbf{e}}_i)}{(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_i)} = (\mathbf{a}, \tilde{\mathbf{e}}_i)$$

Similar to the R^n space, we need to define a functional space which is a set of functions that has operations. All square integrable function is $(a,\, b)$ form a linear space, called the $L^2(a,b)$ space,

$$L^2(a,b) = \left\{ f(x), \quad \int_a^b |f(x)|^2 dx < \infty \right\}. \tag{4.3}$$

It is a linear space because if $f(x) \in L^2(a,b)$ and $g(x) \in L^2(a,b)$, then their linear combination $w(x) = \alpha f(x) + \beta g(x)$ is also in $L^2(a,b)$ for any constant α and β. In $L^2(a,b)$ we can define an ***inner product*** similar to that in R^n space as

$$(f,g) = \int_a^b f(x)\bar{g}(x)dx \tag{4.4}$$

where $\bar{g}(x) = g(x)$ in real number space and is the conjugate of $g(x)$ is complex number space. For example, if $f(x) = e^x + i\sin x$ then $\bar{f}(x) = e^x - i\sin x$.

Example 4.1.

$$f(x) = 1, \qquad g(x) = \sin x, \qquad (f,g) = \int_a^b f(x)\bar{g}(x)dx = \int_a^b \sin x dx = 0$$

We call $f(x)$ and $g(x)$ *orthogonal* (perpendicular) in $L^2(a,b)$ if $(f,\, g) = 0$.

The norm of a function $f(x)$ in $L^2(a,b)$ is defined as

$$\|f\|_2 = \|f\|_{L^2} = \sqrt{(f,f)} = \sqrt{\int_a^b |f(x)|^2 dx}. \tag{4.5}$$

Example 4.2.

$$f(x) = 1, \quad (a,b) = (0,\, 2\pi), \quad \|f\| = \sqrt{\int_0^{2\pi} |f(x)|^2 dx} = \sqrt{2\pi}.$$

Example 4.3.

$$g(x) = 1, \quad (a,b) = (0,\, 2\pi), \quad \|f\| = \sqrt{\int_0^{2\pi} |f(x)|^2 dx} = \sqrt{\pi}.$$

Note that there are many different norms, for example,

$$\|f\|_1 = \int_a^b |f(x)|dx, \quad \|f\|_\infty = \max_{0 \le x \le 2\pi} |f(x)|, \quad \|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p} \tag{4.6}$$

for $p \geq 0$. It can be shown that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

There are more than one ways to define an inner product, so the normal. An inner product is a special functional[2] of two arguments that satisfies

- $(f,\, g) = \overline{(g,\, f)}$
- $(\alpha f + \beta g,\, h) = \alpha(f,\, g) + \beta(g,\, h)$ for any scalars α and β.

A norm should satisfy

- $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f(x) = 0$, or $\int_a^b f^2(x)dx = 0$.
- $\|\alpha f\| = |\alpha|\|f\|$ for any scalar α.
- $\|f + g\| \geq \|f\| + \|g\|$ called the triangle inequality.

All these statements are true in R^n space. The famous Cauchy-Schwartz inequality is true, that is

$$|(f,\, g)| \leq \|f\|\|g\|, \quad \text{or} \quad \left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b f(x)^2dx}\,\sqrt{\int_a^b g(x)^2dx}. \tag{4.7}$$

Particularly, if we take $g(x) = 1$, we get

$$\left|\int_a^b f(x)dx\right|^2 \leq (b-a)\int_a^b f(x)^2dx. \tag{4.8}$$

An example of different inner product is a weighted inner product. Let $w(x) \geq 0$ and $\int_a^b w(x)dx > 0$, the weighted inner product of $f(x)$ and $g(x)$ is

$$(f,\, g)_w = \int_a^b f(x)\overline{g(x)}w(x)dx. \tag{4.9}$$

The function $f(x)$ and $g(x)$ are orthogonal with respect to $w(x)$ on (a, b) if

$$(f,\, g)_w = \int_a^b f(x)\overline{g(x)}w(x)dx = 0. \tag{4.10}$$

The corresponding norm is then

$$\|f\|_w = \sqrt{f,\, f)_w} = \sqrt{\int_a^b w(x)|f(x)|^2dx}. \tag{4.11}$$

We will see the application of weighted inner product and norm for PDEs in polar and spherical coordinates for which $w(r) = r$.

[2] A function whose arguments are functions.

4.2 Expansion of functions in terms of orthogonal set

Definition of an orthogonal set: Let $f_1(x), f_2(x), \cdots, f_n(x), \cdots$ be a set of functions in $L^2(a,b)$, which can also be denoted as $\{f_n(x)\}$. It is called an orthogonal set if $(f_i,\ f_j) = 0$ as long as $i \neq j$ for all i and j's. The orthogonal set is called a ***normal*** orthogonal set if $\|f_i\| = 1$ for all i's.

Example 4.4.

$$f_1(x) = \sin x, f_2(x) = \sin 2x, f_3(x) = \sin 3x, \cdots, f_n(x) = \sin nx, \cdots,$$

or $\{\sin nx\}$ *is an orthogonal set in* $L^2(-\pi,\pi)$.

Proof: Ff $m \neq n$, we can see that

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \int_{-\pi}^{\pi} -\frac{1}{2}\Big(\cos(m+n)x - \cos(m-n)x\Big)\, dx$$

$$= -\frac{1}{2}\left(\frac{\sin(m+n)x}{m+n}\bigg|_{-\pi}^{\pi} + \frac{\sin(m-n)x}{m-n}\bigg|_{-\pi}^{\pi} \right) = 0.$$

Note that if $m = n$, then we have

$$\int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \frac{1-\cos 2mx}{2} dx = \pi. \tag{4.12}$$

Thus we have $\|f_n\| = \sqrt{\pi}$. The new orthogonal set $\left\{\hat{f}_n(x)\right\}$ $=\{f_n(x)/\sqrt{\pi}\}$ is a normal orthogonal set.

Note also that the above discussions are true for any interval $[a, a+2\pi]$ of length of 2π since $\sin nx$ is a period function of 2π.

Function expansion using an orthogonal set.

We can expand a function $f(x)$ using an orthogonal set of functions $\{f_n(x)\}_{n=1}^{\infty}$ that have similar properties as

$$f(x) \sim \sum_{n=1}^{\infty} a_n f_n(x) \tag{4.13}$$

While we can always do this. But the left hand side and right hand side of above may not be the same, and that is why we use the $'\sim'$ sign. To find the coefficients $\{a_n\}_{n=1}^{\infty}$, we assume that the equal sign holds and apply the inner product of the above with a function $a_m(x)$ to get

$$\Big(f(x),\ f_m\Big) = \left(\sum_{n=1}^{\infty} a_n f_n(x), f_m(x)\right) = \sum_{n=1}^{\infty} a_n \Big(f_n(x),\ f_m(x)\Big).$$

Since $\{f_n(x)\}_{n=1}^{\infty}$ is an orthogonal set, the right hand side terms are zeros except the m-th term, that is

$$\Big(f(x),\ f_m\Big) = a_m\Big(f_m(x),\ f_m(x)\Big) \quad\Longrightarrow\quad a_m = \frac{\big(f(x),\ f_m\big)}{\big(f_m(x),\ f_m(x)\big)}. \tag{4.14}$$

Example 4.5. *Expand* $f(x) = x$ *in terms of* $\{\sin nx\}$ *on* $(-\pi, \pi)$.

We know $\{\sin nx\}$ is an orthogonal set on $(-\pi, \pi)$. The coefficient a_n is

$$a_n = \frac{\int_{-\pi}^{\pi} x \sin nx dx}{\int_{-\pi}^{\pi} \sin^2 nx dx} = \frac{1}{\pi}\left(-\frac{x\cos nx}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}dx\right)$$
$$= -\frac{2}{\pi}\frac{\cos nx}{n}.$$

The expansion then is

$$x \sim 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \cdots .$$

From Fourier series theory, we know that the equality sign hold for this case at any x in $(-\pi, \pi)$.

Example 4.6. *Expand* $f(x) = x^2$ *in terms of* $\{\sin nx\}$ *on* $(-\pi, \pi)$.

It is easy to check that $a_n = 0$ for all n's. This is because we have

$$a_n = \frac{\int_{-\pi}^{\pi} x^2 \sin nx dx}{\int_{-\pi}^{\pi} \sin^2 nx dx} = 0.$$

The integrand is an odd function whose integral in symmetric interval is always 0. Such an expansion is meaningless. This is because the function $f(x) = x^2$ does not have many properties of $\{\sin nx\}$ on $(-\pi, \pi)$.

We call the orthogonal set $\{\sin nx\}$ on $(-\pi, \pi)$ is incomplete or subset in the space $L^2(\pi, \pi)$. In Figure 4.1, we show a diagram among functions, $L^2(a, b)$ and $\{\sin nx\}$ and $\{\cos nx\}$ which are subset of $L^(0, \pi)$. While there are not complete in $L^2(0, \pi)$, but they are complete if additional conditions are imposed such as some kind of boundary conditions.

It is easy to check that the set $\{\cos nx\}_{n=0}^{\infty}$ is also an orthogonal set on $(-\pi, \pi)$. Note that this set includes $f(x) = 1$ when $n = 0$. We can expand $f(x) = x^2$ in terms of $\{\cos nx\}_{n=0}^{\infty}$. But it is meaningless to expand $f(x) = x$ in terms of $\{\cos nx\}_{n=0}^{\infty}$. However, if we put the two orthogonal set together to $\{1, \cos nx, \sin nx\}_{n=1}^{\infty}$, then we can show that it is another orthogonal set since $\int_{-\pi}^{\pi} \sin mx \cos nx = 0$ for any m

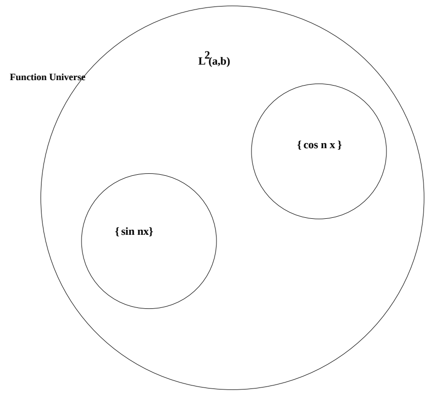

Figure 4.1. *A diagram of the orthogonal functions. If all functions are in a universe. Then $L^2(a,b)$ is a complete subset, called a Hilbert space since an inner product is defined. This set is complete meaning that any Cauchy sequence will converge to a function in $L^2(a,b)$. The orthogonal set $\{\sin nx\}$ and $\{\cos nx\}$ are subsect of $L^2(0,\pi)$.*

and n. Any function $f(x)$ in $L^2(-\pi,\pi)$ can be expanded by the orthogonal set,

$$f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) . \tag{4.15}$$

This is called a Fourier series of $f(x)$ on $(-\pi,\pi)$.

4.3 Sturm-Liouville eigenvalue problems

Sturm-Liouville eigenvalue problems provide a way of generating orthogonal functions that have some special properties. For example, for the 1D wave equations $u_{tt} = c^2 u_{xx}$ in the domain $(0,L)$ with a homogeneous boundary condition $u(0,t) = u(L,t) = 0$. The Sturm-Liouville eigenvalue problem obtained using the method of separation variables would lead to a set of orthogonal functions $\left\{\sin \frac{n\pi x}{L}\right\}$. For any function $f(x) \in L^2(0,L)$ with $f(0) = 0$ and $f(L) = 0$, we can expand the $f(x)$ in terms of the orthogonal functions.

A Sturm-Liouville problem has the following form

$$(p(x)y'(x))' + q(x)y(x) = f(x), \qquad a < x < b \tag{4.16}$$

with two boundary conditions at $x = a$ and $x = b$. Take $x = a$ for example, three liner boundary conditions are often used.

1. The solution is given, that is, $u(a) = \alpha$ is known. It is called a Dirichlet BC.
2. The derivative of the solution is given, that is, $'u(a) = \beta$ is known. It is called a Neumann BC.
3. The BC is given as $\alpha u(a) + \beta u'(a) = \gamma$. It is called a Robin or mixed BC.

We can write a uniform form of the BC at the two ends as

- $c_1 y(a) + c_2 y'(a) = b_1, \qquad c_1^2 + c_2^2 \neq 0.$
- $d_1 y(b) + d_2 y'(b) = b_1, \qquad d_1^2 + d_2^2 \neq 0.$

The ODE is called a self-adjoint ODE. Note that $p(x)y''(x)+w(x)y'(x)+q(x)y(x) = f(x)$ is not a self-adjoint ODE unless it can be transformed to the standard form $(\bar{p}(x)y'(x))' + \bar{q}(x)y(x) = f(x)$.

The Sturm-Liouville problem will have unique solution if $p(x) \geq p_0 > 0$ and $q(x) \leq 0$ with suitable boundary conditions, for example, Dirichlet BC at two ends. However, here we are more interested in multiple solutions

$$\begin{aligned} &\Big(p(x)y'(x)\Big)' + \Big(q(x) + \lambda r(x)\Big) y(x) = 0, \qquad a < x < b, \\ &c_1 y(a) + c_2 y'(a) = 0, \qquad c_1^2 + c_2^2 \neq 0, \\ &d_1 y(b) + d_2 y'(b) = 0, \qquad d_1^2 + d_2^2 \neq 0. \end{aligned} \tag{4.17}$$

This is called a Sturm-Liouville eigenvalue problem. Note that the ODE and the boundary conditions are all homogeneous.

Apparently $y(x) = 0$ is a solution, called a *trivial solution*. We can find some λ such that the problem has non-trivial solution. In a Sturm-Liouville eigenvalue problem, we want to find both λ, and corresponding $y_\lambda(x) \neq 0$ that satisfies both of the ODE and the boundary conditions. We can such $((\lambda, y_\lambda(x))$ an eigenpair.

Example 4.7. *Solve the eigenvalue problem*

$$\begin{aligned} &y'' + \lambda y = 0, \quad 0 < x < \pi, \\ &y(0) = y(\pi) = 0. \end{aligned}$$

Solution: In this example $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. The root of the characteristic polynomial is $\pm\sqrt{-\lambda}$. If $\lambda < 0$, then the solution is

$$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

Plug the boundary conditions $y(0) = y(\pi) = 0$, we get

$$C_1 + C_2 = 0, \qquad C_1 e^{\sqrt{-\lambda}\pi} + C_2 e^{-\sqrt{-\lambda}\pi} = 0.$$

The solution is $C_1 = 0$ and $C_2 = 0$. We have the trivial solution. Similarly if $\lambda = 0$, then $y(x) = C_1 + C_2 x$ and again we get the trivial solution.

However, if $\lambda > 0$, then the solution is

$$y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

The BC $y(0) = 0$ leads to $C_1 = 0$. Thus $y(x) = C_2 \sin\sqrt{\lambda}x$. The second BC $y(\pi) = 0$ leads to $C_2 \sin\sqrt{\lambda}\pi = 0$. When $\sin x = 0$? x bust be $n\pi$. Thus we get

$$\sqrt{\lambda}\pi = n\pi \longrightarrow \lambda = (n\pi)^2, \quad n = 1, 2, \cdots.$$

The solution to the eigenvalue problem is

$$\lambda_n = n^2, \qquad y_n(x) = \sin nx, \qquad n = 1, 2, \cdots.$$

Usually, we do not include the C_2 term since the eigen-functions can differ by a constant. Note that the eigenfunctions $\{\sin nx\}$ is an orthogonal set on $(0, \pi)$.

Class practice

Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad 0 < x < 1,$$
$$y(0) = y(1) = 0.$$

The solution is $\lambda_n = (n\pi)^2$ and $y_n(x) = \sin n\pi x$.

Regular and singular Sturm-Liouville eigenvalue problem

To make the differential equation wellposed, we require $p(x)$ is non-zero. Mathematically we require that $p(x) \in C(a, b)$ and $p(x) \geq p_0 > 0$ and $q(x) \geq 0$ and $r(x) \geq 0$. Such a Sturm-Liouville eigenvalue problem is called a *regular* problem. If the conditions, especially, the condition on $p(x)$ is violated, We called a *singular* problem. Below are some exmples

$$y'' + \lambda y = 0, \quad -1 < x < 1, \quad \text{regular},$$
$$(xy')' + \lambda y = 0, \quad -1 < x < 1, \quad \text{irregular at } x = 0,$$
$$((1 - x^2)y')' + \lambda y = 0, \quad -1 < x < 1, \quad \text{irregular at } x = \pm 1.$$

Sometime, we need some effort to re-write a problem to have the standard Sturm-Liouville eigenvalue problem.

Example 4.8. *$x^2y''+2xy'+\lambda y=0$ can be written as $(x^2y')'+2xy'+\lambda y-2xy'=0$ which is $(x^2y')'+\lambda y=0$.*

Example 4.9. *We can divide by x^2 for $xy''-y'+\lambda xy=0$ to get $\frac{1}{x}y''-\frac{1}{x^2}y'+\lambda y=0$ which is $\left(\frac{1}{x}y'\right)'+\lambda y=0$ which is a standard S-L eigenvalue problem.*

Example 4.10. *Solve the eigenvalue problem*

$$y''+\lambda y=0, \quad 0<x<\pi,$$
$$y(0)=0, \qquad y'(\pi)=0.$$

Solution: From previous example, we know that the solution should be $y(x)=C_2\sin\sqrt{\lambda}x$. Thus the derivative is $y'(x)=C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$. From $y'(\pi)=0$ we get $y'(\pi)=\cos\sqrt{\lambda}\pi=0$. Thus the solution is

$$\sqrt{\lambda}\pi=\frac{1}{2}+n, \quad n=0,1,2,\cdots, \quad \Longrightarrow \quad \lambda_n=\left(\frac{1}{2}+n\right)^2,$$
$$y_n(x)=\sin\left(\frac{1}{2}+n\right)x.$$

Question: Can we take $n=-1,n=-2,\cdots,$?

The set of orthogonal functions $\left\{\sin(\frac{1}{2}+n)x\right\}$ can be used to solve the wave equations $u_{tt}=c^2u_{xx}$ with the boundary condition $u(0,t)=0$ and $\frac{\partial u}{\partial x}(\pi,t)=0$ on the interval $(0,\,\pi)$.

Example 4.11. *Solve the eigenvalue problem with mixed boundary condition*

$$y''+\lambda y=0, \quad 0<x<1,$$
$$y'(0)=0, \qquad y(1)+y'(1)=0.$$

Solution: We know the solution should have the form

$$y(x)=C_1\cos\alpha x+C_2\sin\alpha x, \qquad \lambda=\alpha^2,$$
$$y'(x)=-\alpha C_1\sin\alpha x+\alpha C_2\cos\alpha x.$$

From $y'(0) = 0$, we conclude that $C_2 = 0$. From the mixed boundary condition we have

$$C_1 (\cos\alpha - \alpha \sin\alpha) = 0, \quad \text{or} \quad \cos\alpha - \alpha\sin\alpha = 0, \quad \Longrightarrow \quad \cot\alpha = \alpha.$$

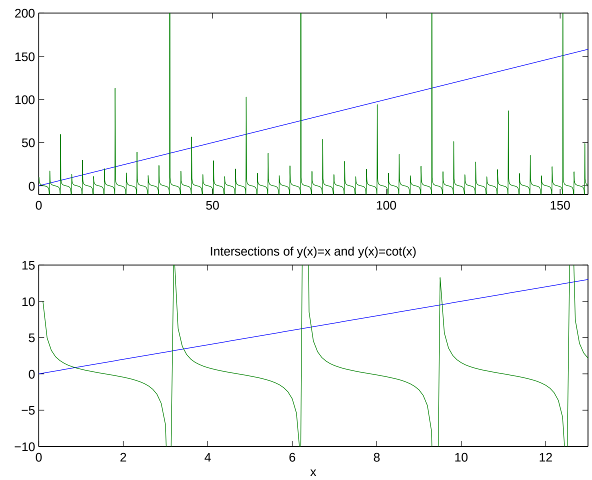

Figure 4.2. *Plot of $y = x$ and $y = \cot x$. The intersections are the eigenvalues of $\sqrt{\lambda_n}$. Note tha some machine errors are present due to the singularities of $\cot x$ at $k\pi$, $k = 1, 2, \cdots$.*

There is no closed form for the solution of the non-linear problem. But we do know the solution is the intersection of $y = x$ and $y = \cot x$. There are infinite positive solutions $\alpha_1 = 0.86\cdots$, $\alpha_2 = 3.43\cdots$, $\alpha_3 = 6.44\cdots$. The eigenvalues are $\lambda_n = \alpha_n^2$, and the eigenfunction is $y_n(x) = \cos\lambda_n x$. In Figure 4.2, we show a plot of $y = x$ and $y = \cot x$. The intersections are α_n's.

4.4 Theory and applications of Sturm-Liouville eigenvalue problem

For a regular Sturm-Liouville eigenvalue problem,

$$\begin{aligned} &\Big(p(x)y'(x) \Big)' + \Big(q(x) + \lambda r(x) \Big) y(x) = 0, \qquad a < x < b, \\ &c_1 y(a) + c_2 y'(a) = 0, \qquad c_1^2 + c_2^2 \neq 0, \\ &d_1 y(b) + d_2 y'(b) = 0, \qquad d_1^2 + d_2^2 \neq 0. \end{aligned} \tag{4.18}$$

The term regular means that all $p(x), q(x), r(x) \in C(a,b)$, $p(x) \geq p_0 > 0$, $r(x) \geq 0$. Then we have the following theorem.

Theorem 4.1.

1. *There are infinite number of eigenvalues, that are all real numbers. We can arrange the eigenvalues as*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{4.19}$$

2. *The eigenfunctions $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x), \cdots$, are an orthogonal set with respect to the weight function $r(x)$ on (a,b), that is*

$$\int_a^b y_m(x) y_n(x) r(x) dx = 0, \qquad if \quad m \neq n. \tag{4.20}$$

3. *For any function $u(x) \in L^2_r(a,b)$ that satisfies the same boundary condition, we have an orthogonal expansion in terms of $\{y_n(x)\}_{n=1}^{\infty}$, that is*

$$u(x) = \sum_{n=1}^{\infty} A_n y_n(x), \qquad with \quad A_n = \frac{\int_a^b u(x) y_n(x) r(x) dx}{\int_a^b y_n^2(x) r(x) dx}. \tag{4.21}$$

Sketch of the proof of the orthogonality: Let $y_k(x)$ and $y_j(x)$ are two different eigenfunctions corresponding to the eigenvalues of λ_k and λ_j respectively. We have

$$\Big(p y_j' \Big)' + \Big(q + \lambda r \Big) y_j = 0, \tag{4.22}$$

$$\Big(p y_k' \Big)' + \Big(q + \lambda r \Big) y_k = 0. \tag{4.23}$$

We multiply (4.23) by $y_j(x)$ and multiply (4.22) by $y_i(x)$ to get

$$y_k \Big(p y_j' \Big)' - y_j \Big(p y_k' \Big)' + (\lambda_j - \lambda_k) r y_j y_k = 0. \tag{4.24}$$

Integrating above from a to b leads to

$$(\lambda_j - \lambda_k) \int_a^b r y_j y_k dx = \int_a^b y_k \left((p y_j')' - y_j (p y_k')' \right) dx.$$

Applying integration by parts to the right hand side and carry out some manipulation, we get

$$(\lambda_j - \lambda_k) \int_a^b r y_j y_k dx = p(b) y_j'(b) y_k(b) - p(a) y_j(b) y_k'(b)$$
$$- p(a) y_j'(a) y_k(a) + p(a) y_j(a) y_k'(a).$$

From the boundary condition at $x = a$ we have

$$\begin{bmatrix} y_j(a) & y_j'(a) \\ y_k(a) & y_k'(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $c_1^2 + c_2^2 \neq 0$, we must have the determined to be zero, that is

$$p(a)y_j'(a)y_k(a) - p(a)y_j(a)y_k'(a) = 0.$$

By the same derivation at $x = b$, we also have

$$p(b)y_j'(b)y_k(b) - p(a)y_j(b)y_k'(b) = 0.$$

Thus we have $(\lambda_j - \lambda_k) \int_a^b r y_j y_k dx = 0$, since $\lambda_j \neq \lambda_k$, we conclude that $\int_a^b r y_j y_k dx = 0$. This completes the proof.

Chapter 5

Method of separation variables for solving PDE BVP in Cartesian coordinates

Consider the boundary value problems

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{5.1}$$

c is called the wave number in physics. We have already known the solution for various situations.

- The general solution $u(x,t) = F(x-ct) + G(x+ct)$.
- Solution to the Cauchy problem $\infty < x < \infty$, $u(x,0) = f(x)$; $\frac{\partial u}{\partial t}(x,0) = g(x)$, the D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2c}\int_{x-at}^{x+at} g(s)ds.$$

- The solution to the IvP-BVP problems $0 < x < L$, the normal modes solution for some special $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$,

$$f(x) = \sum_{n=1}^{n} a_n \sin\frac{n\pi x}{L}, \qquad g(x) = \sum_{n=1}^{n} b_n \sin\frac{n\pi x}{L}.$$

  The solution would be

$$u(x,t) = \sum_{n=1}^{n}\left(a_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + \frac{b_n}{n\pi c}\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\right).$$

Challenging: How about different boundary condition $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(L,t) = 0$. What are the normal modes solutions?

5.1 Series solution of 1D wave equations of IVP-BVP

We use the method of separation variables to solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L$$

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

$$u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L,$$

for general $f(x)$ and $g(x)$. Note the consistency requires that $f(0) = u(0,0) = 0$. The method of separation variables includes the following steps.

Step 1: Let $u(x,t) = T(t)X(x)$ and plug its partial derivatives to the original PDE so that we can separate variables. The homogeneous boundary conditions require $X(0) = X(L) = 0$. Differentiating with $u(x,t) = T(t)X(x)$ with t and x respectively, we get

$$\frac{\partial u}{\partial t} = T'(t)X(x), \quad \frac{\partial^2 u}{\partial t^2} = T''(t)X(x); \qquad \frac{\partial u}{\partial x} = T(t)X'(x), \quad \frac{\partial^2 u}{\partial x^2} = T(t)X''(x).$$

The wave equation can be re-written as

$$T''(t)X(x) = c^2 T(t)X''(x), \Longrightarrow \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{5.2}$$

This is because in the last equality, the left hand is a function of t while the right hand side is a function of x, which is possible only both of them are a constant independent of t and x. We get an eigenvalues either for $X(x)$ or $T(t)$. Since we know the boundary condition for $X(x)$, naturally we should solve

$$\frac{X''(x)}{X(x)} = \lambda \quad \text{or} \quad X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0 \tag{5.3}$$

first.

Step 2: Solve the eigenvalue problem. From the Sturm-Liouville eigenvalue theory, we know that $\lambda > 0$, thus the solution is

$$X''(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x.$$

From the boundary condition $X(0) = 0$, we get $C_1 = 0$. From the boundary condition $X(L) = 0$, we get

$$C_2 \sin \sqrt{\lambda} L = 0, \quad \Longrightarrow \quad \sqrt{\lambda} L = n\pi, \quad n = 1, 2, \cdots,$$

since $C_2 \neq 0$. The eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, \cdots, .$$

Now we solve for $T(t)$ using

$$T''(t) + c^2\lambda_n T(t) = 0. \tag{5.4}$$

The solution is (not an eigenvalue problem anymore since we have already known λ_n)

$$T_n(t) = b_n \cos\frac{cn\pi t}{L} + b_n^* \sin\frac{cn\pi t}{L}.$$

Put $X_n(x)$ and $T_n(t)$ together, we get a normal mode solution

$$u_n(x,t) = \sin\frac{n\pi x}{L}\left(b_n \cos\frac{cn\pi t}{L} + b_n^* \sin\frac{cn\pi t}{L}\right), \tag{5.5}$$

which satisfy the PDE, the boundary conditions, but not the initial conditions.

Step 3: Put all the normal solution together to get the series solution. The coefficients are obtained from the orthogonal expansion of the initial conditions.

The solution to the IVP-BVP of the 1D wave equation can be written as

$$u(x,t) = \sum_{n=0}^{\infty} \sin\frac{n\pi x}{L}\left(b_n \cos\frac{cn\pi t}{L} + b_n^* \sin\frac{cn\pi t}{L}\right) \tag{5.6}$$

which satisfies the PDE and the boundary conditions. The coefficients of b_n and b_n^* are determined from the initial conditions $u(x,0)$ and $u_t(x,0)$.

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad \Longrightarrow \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} \sin\frac{n\pi x}{L}\left(-b_n\frac{cn\pi}{L}\sin\frac{cn\pi t}{L} + b_n^*\frac{cn\pi}{L}\cos\frac{cn\pi t}{L}\right),$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \sin\frac{cn\pi t}{L} b_n^*\frac{cn\pi}{L}, \qquad \Longrightarrow \qquad b_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

Example 5.1. *Solve the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1,$$

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Solution: In this example, $c = 1$, $L = 1$, and $g(x) = 0$, we have $b_n^* = 0$ and

$$b_n = 2\int_0^{\frac{1}{2}} f(x)\sin n\pi x dx = 2\int_0^{\frac{1}{2}} \sin n\pi x dx = -\frac{2}{n\pi} \quad \cos n\pi\Big|_0^{\frac{1}{2}}$$
$$= -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right) = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right).$$

It is the half range sine expansion of $f(x)$. The solution to the IVP-BVP of the PDE is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\sin n\pi x \cos n\pi t.$$

We know the series is convergent in the interval (0, 1).

Example 5.2. *Solve the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1,$$

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = \sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

For this example, we can use the normal modes solution

$$u(x,t) = \sin\pi x\cos\pi ct - \frac{1}{2}\sin 2\pi x\cos 2\pi ct + \frac{1}{3}\sin 3\pi x\cos 3\pi ct.$$

5.2 Series solution of 1D heat equations of IVP-BVP

We use the method of separation variables to solve

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L$$
$$u(0,t) = 0, \qquad u(L,t) = 0.$$
$$u(0,t) = f(x), \qquad 0 < x < L,$$

for a general $f(x)$. Note the consistency requires that $f(0) = u(0,0) = 0$. The method of separation variables includes the following steps.

Step 1: Let $u(x,t) = T(t)X(x)$ and plug its partial derivatives to the original PDE so that we can separate variables. The homogeneous boundary conditions require $X(0) = X(L) = 0$. Differentiating with $u(x,t) = T(t)X(x)$ with t and x respectively, we get

$$\frac{\partial u}{\partial t} = T'(t)X(x); \qquad \frac{\partial u}{\partial x} = T(t)X'(x), \quad \frac{\partial^2 u}{\partial x^2} = T(t)X''(x).$$

The wave equation can be re-written as

$$T'(t)X(x) = c^2 T(t)X''(x), \Longrightarrow \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{5.7}$$

This is because in the last equality, the left hand is a function of t while the right hand side is a function of x, which is possible only both of them are a constant independent of t and x. We get an eigenvalues either for $X(x)$ or $T(t)$. Since we know the boundary condition for $X(x)$, naturally we should solve

$$\frac{X''(x)}{X(x)} = \lambda \quad \text{or} \quad X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0 \tag{5.8}$$

first.

Step 2: Solve the eigenvalue problem. From the Sturm-Liouville eigenvalue theory, we know that $\lambda > 0$, thus the solution is

$$X''(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

From the boundary condition $X(0) = 0$, we get $C_1 = 0$. From the boundary condition $X(L) = 0$, we get

$$C_2 \sin\sqrt{\lambda}L = 0, \quad \Longrightarrow \quad \sqrt{\lambda}L = n\pi, \quad n = 1, 2, \cdots,$$

since $C_2 \neq 0$. The eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \cdots, .$$

Now we solve for $T(t)$ using

$$T'(t) + c^2 \lambda_n T(t) = 0. \tag{5.9}$$

The solution is (not an eigenvalue problem anymore since we have already known λ_n)

$$T_n(t) = b_n e^{c^2 \lambda_n t} = b_n e^{-c^2 (\frac{n\pi}{L})^2 t}.$$

Put $X_n(x)$ and $T_n(t)$ together, we get a normal mode solution

$$u_n(x,t) = \sin\frac{n\pi x}{L} b_n e^{-c^2 (\frac{n\pi}{L})^2 t}, \tag{5.10}$$

which satisfy the PDE, the boundary conditions, but not the initial conditions.

Step 3: Put all the normal solution together to get the series solution. The coefficients are obtained from the orthogonal expansion of the initial conditions.

The solution to the IVP-BVP of the 1D wave equation can be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-c^2 (\frac{n\pi}{L})^2 t} \tag{5.11}$$

which satisfies the PDE and the boundary conditions. The coefficients of b_n are determined from the initial conditions $u(x,0)$,

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad \Longrightarrow \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Chapter 6

Various Fourier series, properties and convergence

We have seen that $\left\{\sin\frac{n\pi x}{L}\right\}$ and $\left\{\cos\frac{n\pi x}{L}\right\}$ play very an important role in the series of solution of PDEs using the method of separation variables. While these orthogonal functions are obtained from Sturm-Liouville eigenvalue problems, they should have remind us of Fourier series in which $\{\sin nx\}$ and $\{\cos nx\}$ are used. Fourier series have wide applications in electro-magnetics, signal processing, filter design and many areas in engineering. In this chapter, we will introduce various Fourier series, discuss the properties and convergence of those series, and the relation to some of PDE solutions of IVP-BVP.

We will see three kind of Fourier expansions of a function $f(x)$.

1. General Fourier expansions in $(-L, L)$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right), \tag{6.1}$$

   when $L = \pi$, we get the classical Fourier series.

2. Half-range sine expansions

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{6.2}$$

3. Half-range cosine expansions

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}. \tag{6.3}$$

6.1 Period, piecewise continuous/smooth functions

We know that $\sin x, \cos x, \sin 2x, \cos 2x, \cdots$ are all period functions. What is a period function? A function repeats itself in a fixed pattern.

Definition 6.1. *If there is a positive number T such at $f(x+T) = f(x)$ for any x, then $f(x)$ is called a period function with a period T.*

According to the definition, $f(x)$ should be defined in the entire space $(-\infty, \infty)$. Also, if $f(x) = f(x+T)$, then $f(x+2T) = f(x+T+T) = f(x+T) = f(x)$, thus $2T$ is also a period of $f(x)$. To avoid the confusion, we only use the smallest $T > 0$ which is called the fundamental period, or simply the period, for short.

Example 6.1. *Find the period of* $\sin x, \cos x, \tan x, \cot x$.

The period of $\sin x, \cos x$ is 2π, while the period of $\tan x, \cot x$ is π.

Example 6.2. *Are the following period functions? If so, find the period of the functions.*

$$\cos \pi x, \ \sin x + \tan x, \ \sin x + \cos \frac{x}{2}, \ \sin x + x, \cos mx.$$

1. Yes, $\cos \pi x = \cos \pi(x+T) = \cos(\pi x + \pi T)$, we get $T + 2$.
2. Yes, the sum of two periodic functions is still periodic, the period is the larger one, $T = 2\pi$.
3. Yes, $\sin x + \cos \frac{x}{2} = \sin(x+T) + \cos \frac{x+T}{2}$. Since the period of the second function is $T/2 = 2\pi$. We know the period is $T = 4\pi$.
4. No, since x is not a periodic function.
5. Yes, $\cos mx = \cos m(x+T) = \cos(mx + mT)$. Thus $mT = 2\pi$ and $T = \dfrac{2\pi}{m}$.

Example 6.3. *Let $f(x) = x - int(x) = x - [x]$, where $[x]$ is called a floor function, $[x]$ = greatest integer not larger than x, for example, $[1.5] = 1$, $[0.5] = 0$, $[-1.5] = -2$, or the integer toward left. Then $f(x)$ is a period function with period $T = 1$ that can be expressed as*

$$f(x) = x, \qquad \text{if } \ 0 \le x < 1, \tag{6.4}$$

and $f(x) = f(x+1)$. It is enough to write down the expression in one period, see Figure 6.3 (b).

Example 6.4. *The sawtooth function is determined by*

$$f(x) = \begin{cases} \frac{1}{2}\left(-\pi - x\right) & \text{if } -\pi \le x \le 0, \\ \frac{1}{2}\left(\pi - x\right) & \text{if } 0 < x \le \pi, \end{cases} \tag{6.5}$$

and $f(x) = f(x+2\pi)$*, see Figure 6.3 (a). However, it would be easier if we use the expression in the interval* $(0, 2\pi)$ *since it is a continuous piece as*

$$f(x) = \frac{1}{2}\left(\pi - x\right), \qquad 0 < x \le 2\pi,$$

and $f(x) = f(x+2\pi)$*.*

Piecewise continuous/smooth functions

If a function $f(x)$ is continuous in $[a,\, b]$, then for any $x_0 \in [a,\, b]$, we have $\lim\limits_{x\to x_0} f(x) = f(x_0)$. We call that $f(x) \in C[a,b]$. It is obvious that $f(x) = \sin x, \cos, x^3 + 1$ and their linear combinations are continuous function in any interval $[a,b]$. The functions $f(x) = 1/x$ is discontinuous at $x = 0$, or any interval that contains zero. Note that it is continuous on (0, 1) but not $[0,1]$. The function $\tan x$ is continuous on $[0,1]$ but not on $[0, \frac{\pi}{2}]$ since $\lim\limits_{x\to\frac{\pi}{2}-} f(x) = \infty$.

If there are finite number of points $x_1, x_2, \cdots, x_N$ in $[a,\, b]$ at which the function is not continuous but

$$\lim_{x\to x_i-} f(x) = f(x_i-) \;\;\&\;\; \lim_{x\to x_i+} f(x) = f(x_i+) \text{ exist but } \quad f(x_i-) \ne f(x_i+).$$

Such a function is called a piecewise continuous function in (a,b), or precisely, a piecewise continuous and bounded function. Below is an example of a piecewise continuous and bounded function.

Example 6.5. *The Heaviside function* $H(x) = \begin{cases} 0 & \text{if } -\infty < x < 0, \\ 1 & \text{if } 0 \le x < \infty, \end{cases}$

Example 6.6. *The hat function* $h(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{otherwise}, \end{cases}$

If $f'(x)$ is continuous on (a,b), then $f(x)$ is called a smooth function on (a,b). If the derivative $f'(x)$ is a piecewise function, then the function is called a *piecewise smooth function.* The hat function is a piecewise smooth function since.

Example 6.7. *The hat function* $h'(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ 1 & \text{if } -1 < x < 0, \\ -1 & \text{if } 0 < x < 1. \end{cases}$

Properties of period functions

The set of all period functions with the same period T form a linear space. That is, let $f(x)$ and $g(x)$ be two period functions of period T, then $w(x) = \alpha f(x) + \beta g(x)$ is also a period function of period T. Note again that a period function is defined in the entire space $-\infty < x < \infty$.

Theorem 6.2. *Let $f(x)$ be a period function of period T, then*

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx \tag{6.6}$$

for any real number a.

Proof: If the theorem is true, then $\int_a^{a+T} f(x)dx$ is a constant as a function of a, thus we define $F(a) = \int_a^{a+T} f(x)dx$, then

$$\frac{dF(a)}{da} = f(a+T) - f(a) = 0.$$

Thus $F(a)$ is a constant, $F(0) = F(a) = F(-T/2) = \cdots$

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx = \int_{a-\frac{T}{2}}^{a+\frac{T}{2}} f(x)dx.$$

Often we prefer to use the period that

- $f(x)$ is a continuous piece.
- integration starts from the origin ($a = 0$).
- integration from a symmetric interval $(-\frac{T}{2}, \frac{T}{2})$.

6.2 The classical Fourier series expansion and partial sums

Let $f(x)$ be a periodic function of 2π and in $L^2(-\pi, \pi)$, then its classical Fourier series expansion is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{6.7}$$

The coefficients $\{a_n\}$ and $\{b_n\}$ are called the Fourier coefficients are determined from

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \tag{6.8}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \qquad n = 1, 2, \cdots . \tag{6.9}$$

Applications includes

- Express $f(x)$ in terms of simpler functions.
- Provide an approximation method for evaluating $f(x)$ using the partial summer defined as

  $$S_N(x) = a_0 + \sum_{n=1}^{N}\left(a_n \cos nx + b_n \sin nx\right). \tag{6.10}$$

  as used in many computer packages for a given number N. We hope that $\lim_{N\to\infty} S_N(x) = f(x)$.
- Basis for several fast algorithms such as Fast Fourier Transform (FFT)
- Used for spectral (frequency) analysis, signal processing, filters, etc.

Note that if x is a time variable for some physical applications, we call that $f(x)$ is defined in the time domain, while $\left\{\sqrt{a_n^2 + b_n^2}\right\}_0^\infty$ with $b_0 = 0$, which is defined in the frequency domain.

Example 6.8. *Find the classical Fourier series of $f(x) = x$.*

Solution: Note that we are expanding a period function (truncated and extended function)

$$\tilde{f}(x) = \begin{cases} x & \text{if } |x| \le \pi, \\ \tilde{f}(x + 2\pi) & \text{otherwise.} \end{cases}$$

The function is piecewise continuous and bounded with discontinuities at $x = 2n\pi, n = 1, 2, \cdots,$. We carry out the following to determine the coefficients

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0, \quad n = 1, 2, \cdots,$$

since $f(x)$ and $f(x)\cos nx$ are odd functions. Furthermore, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin nx dx = -\frac{2}{n\pi} x \cos nx \Big|_0^\pi = (-1)^{n+1}\frac{2}{n\pi}$$

Thus we get

$$x \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin nx = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \cdots .$$

From the plots of the Maple, we can see that the partial sum $S_N(x)$

1. converges to $f(x)$ in the interior of (π, π) as $N \to \infty$;
2. does not 'converge' at $x = \pm\pi$, but
$$\lim_{N\to\infty} S_N(x) = \frac{\tilde{f}(\pi-) + \tilde{f}(\pi+)}{2}; \tag{6.11}$$
3. $S_N(x)$ oscillates at the discontinuities $\pm 2n\pi$. It is called the Gibbs phenomenon.

In Figure 6.1, we plot the function $f(x) = x$ and a few partial sums $S_1(x)$, $S_5(x)$, $S_{55}(x)$.

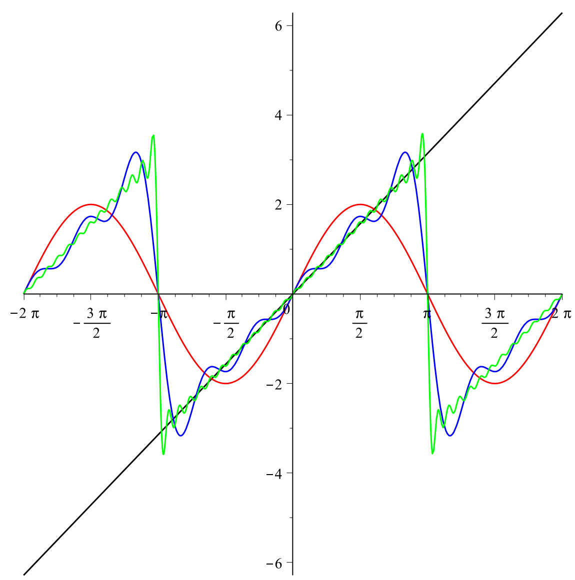

Figure 6.1. *Plot of the Fourier series of x and some partial sums $S_1(x)$, $S_5(x)$, $S_{55}(x)$.*

Example 6.9. *Find the classical Fourier series of $f(x) = 10\sin x + 5\sin 6x + \frac{1}{2}\cos 30x$.*

Solution: The Fourier series of $f(x)$ is itself with $a_{30} = 0.5$, $b_1 = 10$, $b_6 = 5$. In Figure 6.2, we show the plot of the function and see clearly the three frequencies and their strengths.

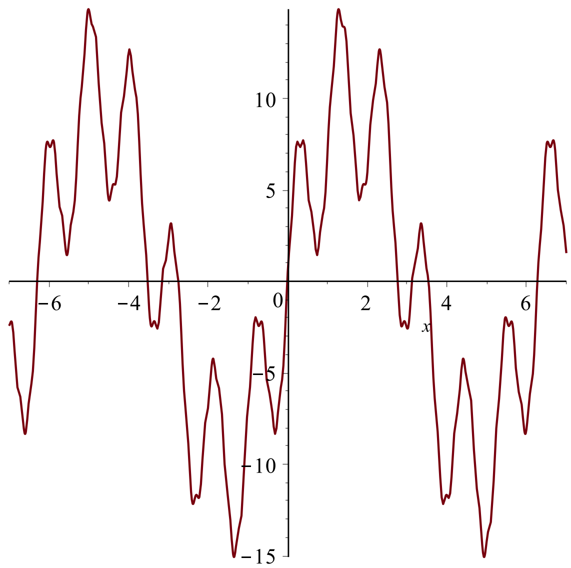

Figure 6.2. *Plot of* $f(x) = 10\sin x + 5\sin 6x + \frac{1}{2}\cos 30x$. *We can see clearly three different frequencies.*

Example 6.10. *The Fourier series of the sawtooth function* $f(x) = \frac{1}{2}(\pi - x)$, $f(x + 2\pi) = f(x)$.

Note that $f(x)$ is an add function in the interval of $(-\pi, \pi)$, thus $a_n = 0$, for $n = 0, 1, \cdots,$. For the coefficients of b_n, it is easier to use one continuous piece in $(0, 2\pi)$, thus

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx dx \\
&= \frac{1}{2\pi}\left(\int_0^{2\pi} \pi \sin nx dx + \frac{x\cos nx}{n}\Big|_0^{2\pi} - \int_0^{2\pi} \frac{x\cos nx}{n} dx\right) \\
&= \frac{1}{2\pi}\frac{2\pi}{n} = \frac{1}{n}.
\end{aligned}
$$

Thus the Fourier series is, see also Figure 6.3,

$$\frac{1}{2}(x-\pi) \sim \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + c \cdots + \frac{\sin nx}{n} + \cdots .$$

We observe that discontinuities are at $x = 2n\pi$.

(a) (b)

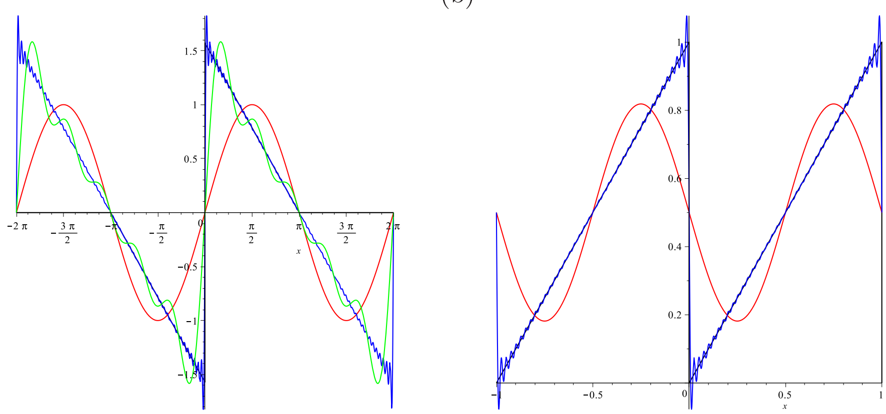

Figure 6.3. *The periodic function and some partial sum plots of the Fourier series (a), the sawtooth function; (b) the floor function.*

Example 6.11. *The Fourier series of the triangular wave.*

$$f(x) = \begin{cases} x+\pi & \text{if } -\pi \le x < 0, \\ \pi - x & 0 \le x \le \pi, \end{cases} \tag{6.12}$$

and $f(x) = f(x+2\pi)$*. Note that we can rewrite* $f(x) = \pi - |x|$ *in the interval* $(-\pi, \pi)$*, which is an even function in* $(-\pi, \pi)$*.*

Solutions: Note that $f(x)$ is an even function, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)dx = -\frac{1}{\pi}\frac{(\pi-x)^2}{2}\Big|_0^{\pi} = \frac{1}{2\pi}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx dx \\
&= \frac{2}{\pi}\left(\frac{(\pi - x)\sin nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx dx\right) \\
&= \frac{2}{\pi}\left(\frac{-\cos nx}{n^2}\Big|_0^{\pi}\right) = \frac{2}{\pi}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right) \\
&= \frac{2}{\pi}\begin{cases} \dfrac{2}{n^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.}\end{cases}
\end{aligned}$$

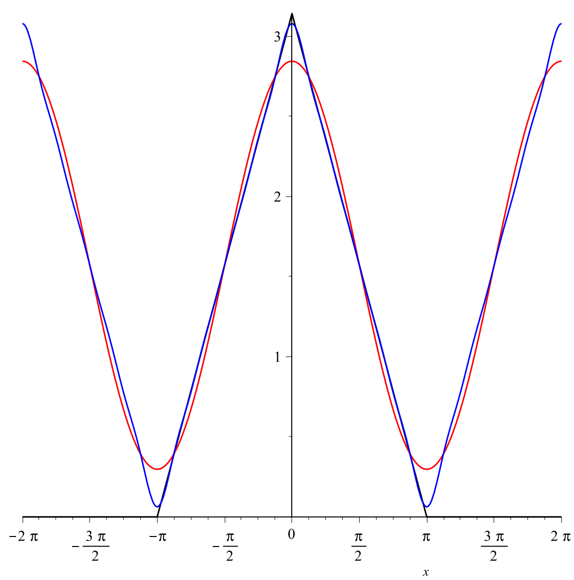

Figure 6.4. *Plot of the triangular wave and several partial sums. There is no Gibbs phoenomenon.*

We can use one simple notation $a_{2k+1} = \dfrac{4}{\pi(2k+1)^2}$. Thus we have, see also Figure 6.4,

$$f(x) = \frac{1}{2\pi} + \sum_{n=0}^{\infty}\frac{4}{\pi(2n+1)^2}\cos\pi(2n+1)x.$$

Since $f(x)$ is continuous everywhere, we have the equality! We can get some identities from Fourier series. In this example, we have

$$f(0) = \pi = \frac{1}{2\pi} + \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)^2}. \tag{6.13}$$

We get

$$\frac{\pi^2}{2} = \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2} \quad \Longrightarrow \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{7^2} + \cdots .$$

6.3 Fourier series of functions with arbitrary periods

Given a period function $f(x) \in L^2(-L, L)$ with $f(x+T) = f(x)$ and $T = 2L$, we can also have a Fourier series expansion of $f(x)$ in $(-L,\ L)$. The idea is to use a linear transform to convert the interval $(-L, L)$ to $(-\pi\,\pi)$ to apply the Fourier expansion.

Let $t = \alpha x$ and α is chosen such that when $x = -L$, $t = -\pi$, and when $x = L$, $t = \pi$. Thus we get $\alpha = \frac{\pi}{L}$. Define also $f(x) = f(\frac{t}{\alpha}) = F(t)$. We can verify that $F(t)$ is a period function of 2π since

$$F(t+2\pi) = f\left(\frac{t+2\pi}{\alpha}\right) = f\left(\frac{t}{\alpha} + \frac{2\pi}{\alpha}\right) = f\left(\frac{t}{\alpha} + 2L\right) = F(t).$$

Thus $F(t)$ has the Fourier series

$$F(t) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right),$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)dt, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos nt\, dt,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin nt\, dt, \qquad n = 1, 2, \cdots .$$

By changing the variable again using $t = \frac{\pi}{L}$ in all the expressions above, we get

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \qquad n = 1, 2, \cdots .$$

Again the partial sum of the Fourier expansion in $(-L, L)$ is defined as

$$S_N(x) = a_0 + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{6.14}$$

for a positive integer $N > 0$.

Example 6.12. *Expand $f(x) = |x|$ in Fourier series in $(-p, p)$ for a parameter $p > 0$. Note that $f(x)$ is an even function, we have*

$$\begin{aligned}
a_0 &= \frac{1}{2p} \int_{-p}^{p} |x| dx = \frac{2}{2p} \int_0^p x dx = \frac{1}{p} \left. \frac{x^2}{2} \right|_0^p = \frac{p}{2}, \\
a_n &= \frac{1}{p} \int_{-p}^{p} |x| \cos \frac{n\pi x}{p} dx = \frac{2}{2p} \int_0^p x \cos \frac{n\pi x}{p} dx \\
&= -\frac{2p}{(n\pi)^2} (1 - \cos n\pi) = \begin{cases} 0 & \text{if } n = 2k, \\ -\dfrac{4p}{(n\pi)^2} & \text{if } n = 2k+1, \end{cases} \\
b_n &= \frac{1}{p} \int_{-p}^{p} |x| \sin \frac{n\pi x}{p} dx = 0,
\end{aligned}$$

since $f(x)$ and $f(x) \cos \frac{n\pi x}{p}$ are even functions, and $f(x) \sin \frac{n\pi x}{p}$ is an odd function. Thus we get

$$\begin{aligned}
|x| &= \frac{p}{2} - \sum_{n=0}^{\infty} \frac{4p}{((2n+1)\pi)^2} \cos \frac{(2n+1)\pi x}{p} \\
&= \frac{p}{2} - \frac{4p}{\pi^2} \left(\cos \frac{\pi x}{p} + \frac{1}{3^2} \cos \frac{3\pi x}{p} + \frac{1}{5^2} \cos \frac{5\pi x}{p} + \frac{1}{7^2} \cos \frac{7\pi x}{p} + \cdots \right).
\end{aligned}$$

In Figure 6.5, we take $p = 1$ and plot the Fourier series and several partial sums of $|x|$ in the interval $(-2, 2)$. The Fourier series converges to $|x|$ only in the interval $(-1, 1)$. No Gibbs phenomenon is presented since the function is piecewise smooth. But we do see that the partial sums smooth the kink of $|x|$ at $x = 0$.

When $p = \pi$, we get the classical Fourier series in $(-\pi, \pi)$,

$$\begin{aligned}
|x| &= \frac{\pi}{2} - \sim \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \cos(2n+1)x \\
&= \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \frac{1}{7^2} \cos 7x + \cdots \right).
\end{aligned}$$

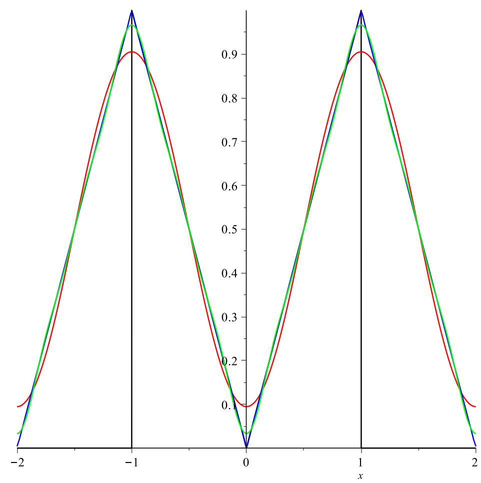

Figure 6.5. *Plot of the Fourier series and several partial sums of $|x|$ in the interval $(-2, 2)$. The Fourier series converges to $|x|$ only in the interval $(-1, 1)$.*

Remark 6.1. *In the expansion above, we expand the $2p$-function $f(x) = |x|$ and $f(x + 2p) = f(x)$ in terms of the Fourier series. The expansion is the same as function $g(x) = |x|$ in the interval $(-p, p)$ but totally different outside the interval. There is no Gibbs phenomenon and the the series is convergent to $|x|$ in $(-p, p)$. The process can be summarised as extension and expansion.*

Example 6.13. *Expand $f(x) = \sin x$ in Fourier series in $(-p, p)$ for a parameter $p > 0$.*

If $p = \pi$, then the Fourier expansion of $\sin x$ is itself. Otherwise, we can expand $\sin x$ in terms of $\sin \frac{n\pi x}{p}$. Note that $a_n = 0$, $n = 0, 1, \cdots$, since $f(x)$ is an odd function. Thus we get

$$\begin{aligned} b_n &= \frac{1}{p}\int_{-p}^{p} \sin x \sin \frac{n\pi x}{p} dx = \frac{2}{2p}\int_{0}^{p} \sin x \sin \frac{n\pi x}{p} dx \\ &= \frac{2p\,(n\pi \sin p\, \cos n\pi - p\cos p\, \sin(n\pi))}{p^2 - \pi^2 n^2}. \end{aligned}$$

The integration is obtained by using the formula

$$\sin\alpha \sin\beta = -\frac{1}{2}\left(\cos(\alpha+\beta) - \cos(\alpha-\beta)\right)$$

or using the Maple command

```
\int_0^p \sin x \sin \frac{n \pi x}{p} dx;
```

For the special case $p = 1$, we can get

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n\pi \sin 1 \sin(n\pi x)}{n^2\pi^2 - 1}.$$

which is valid only in the interval $(-1, 1)$, see Figure 6.6 for an illustration.

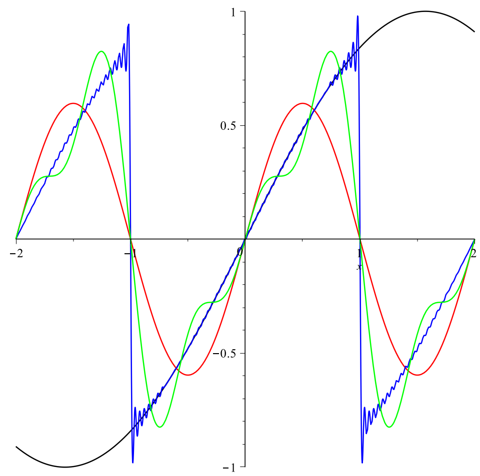

Figure 6.6. *Plot of the Fourier series and several partial sums of* $\sin x$ *in the interval* $(-2, 2)$*. The Fourier series converges to* $\sin x$ *only in the interval* $(-1, 1)$*.*

6.4 Half-range expansions

In this section, we can see that we can choose different ways of expansions and see some connections between Fourier series and orthogonal functions from Sturm-Liouville eigenvalue problems. With half-range expansion, we can also reduces some work load compared to a full range expansion, and impose some special properties of the expansions. The techniques once again is based some particular extensions and expansions.

Let $f(x)$ be a piecewise smooth function in $(0, L)$[3]. If we extend $f(x)$, $0 \leq$

[3]In fact, the discussions work for any interval $(a,\, b)$ $(b > a)$. we can use a shift $s = x - a$ to change the domain from $(a,\, b)$ in x to $(0,\, b - a)$ in terms of s.

$x \leq L$ in the following way,

$$f_e(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L, \\ f(-x) & \text{if } -L < x < 0, \end{cases} \tag{6.15}$$

then we can have the Fourier series expansion of $f_e(x)$ in the interval $(-L, L)$. Since $f_e(x)$ is an even function, we have $b_n = 0$ and the expansion has cosine functions only

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_e(x)dx = \frac{1}{L}\int_0^L f(x)dx, \tag{6.16}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{6.17}$$

Also in the interval, we have $f_e(x) = f(x)$, thus we obtain

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \tag{6.18}$$

which is called the half-range cosine series expansion of $f(x)$.

Similarly, if we extend $f(x)$, $0 \leq x \leq L$ according to

$$f_o(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L, \\ -f(-x) & \text{if } -L < x < 0, \end{cases} \tag{6.19}$$

then we can have the Fourier series expansion of $f_e(x)$ in the interval $(-L, L)$. Since $f_o(x)$ is an odd function, we have $a_n = 0$ and the expansion has sine functions only

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{6.20}$$

Also in the interval, we have $f_o(x) = f(x)$, thus we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{6.21}$$

which is called the half-range sine series expansion of $f(x)$.

Example 6.14. *Expand $f(x) = x$ in both half range cosine and sine series in $(0, 1)$. What is the relation of the expansion with the Fourier series in $(-1, 1)$.*

Solution: The function $f(x)$ is an odd function, thus the half range sine series is the same as the Fourier series in $(-1, 1)$. For the sine expansion, we have (verified by Maple)

$$b_n = 2\int_0^1 x\sin(n\pi x)dx = -2\frac{\cos n\pi}{n\pi} = (-1)^n\frac{2}{n\pi}$$

Thus the sine expansion (and the Fourier expansion) of $f(x) = x$ in the interval $(0, 1)$ is

$$x = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n\pi}\sin(n\pi x).$$

The series is convergent in the interior of $[0, 1)$ but not at $x = 1$, see Figure 6.7 (a) for the plots of the expansion and several partial sums.

For the cosine half range expansion, the expansion is only in $(0, 1)$, we have

$$a_0 = \int_0^1 x dx = \frac{1}{2} \qquad a_n = 2\int_0^1 x\cos(n\pi x)dx = 2\frac{\cos n\pi - 1}{(n\pi)^2} = \frac{-4}{(2k-1)^2\pi^2}.$$

Thus the cosine expansion of $f(x) = x$ in the interval $(0, 1)$ is

$$x = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2}.$$

The series is convergent in the entire interval of $[0, 1]$, see Figure 6.7 (a) for the plots of the expansion and several partial sums. In this case, we have fast convergence of the partial sum of the cosine expansion.

(a) (b)

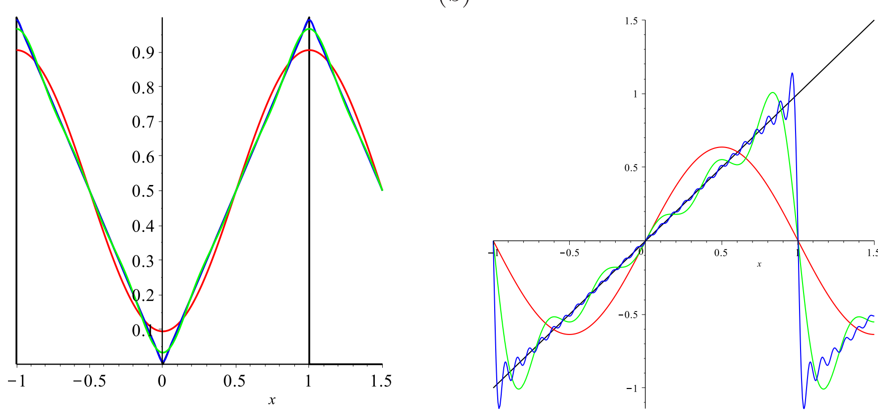

Figure 6.7. *Half range sine/cosine Fourier expansions of $f(x) = x$ in $(0, 1)$ and several partial sums. (a) Half cosine, the series is convergent in $[0, 1]$; (b) Half sine, the series is convergent in $[0, 1)$ but not at $x = 1$.*

Example 6.15. *Expand $f(x) = \cos x$ in both half range cosine and sine series in $(0, \pi)$. What is the relation of the expansion with the Fourier series in $(-\pi, \pi)$. How about in $(0, 1)$?*

Solution: The function $f(x) = \cos x$ is an even function, thus the half range cosine series is the same as the Fourier series in $(-\pi, \pi)$ or any 2π intervals, which is itslef but it is different in $(-1, 1)$.

For the sine expansion, we have (verified by Maple)

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(nx)dx = \frac{2}{\pi}\frac{n(\cos n\pi + 1)}{n^2 - 1} = \frac{1}{\pi}\frac{8k}{(2k)^2 - 1}.$$

Thus the sine expansion of $f(x) = \cos x$ in the interval $(0, \pi)$ is

$$\cos x = \sum_{n=1}^{\infty} \frac{1}{\pi}\frac{8n}{4n^2 - 1}\sin(2nx).$$

The series is convergent in the interior of $[0, \pi)$ but not at $x = 1$, see Figure 6.7 (a) for the plots of the expansion and several partial sums. We also plot the cosine expansion and several partial sums of $\cos x$ in $(0, 1)$. In this case, the series is convergent in the entire interval ($[0, 1]$ including two end points.

(a) (b)

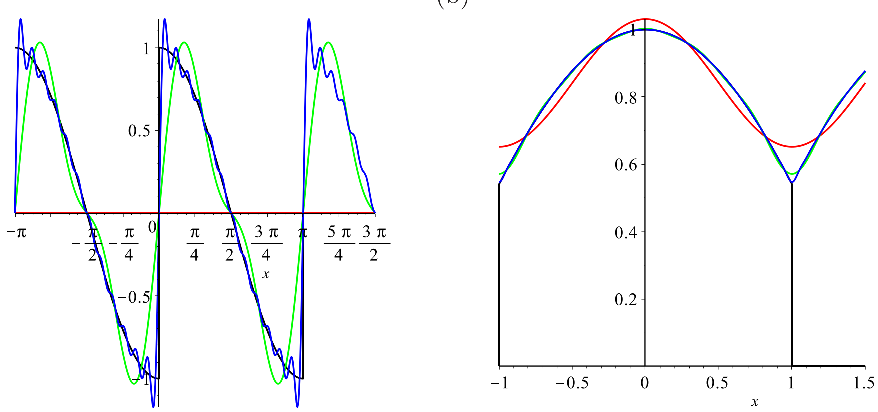

Figure 6.8. *Half range sine/cosine Fourier expansions of $f(x) = \cos x$ and several partial sums. (a) Half sine in $(0, \pi)$, the series is convergent in $(0, \pi)$ but not at two ends; (b) Half cosine in $[0, 1]$, the series is convergent in $[0, 1]$ including two ends.*

6.5 Some theoretical results of Fourier type of series

First of all, from the orthogonality of $\left\{\cos\frac{n\pi x}{L}\right\}_0^\infty$ and $\left\{\sin\frac{n\pi x}{L}\right\}_1^\infty$, we can easily prove the Parseval's identity.

Parseval's identity If $f(x) \in L^2(-L, L)$ and

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L},\right), \quad -L < x < L,$$

then the following Parseval's identity holds

$$\frac{1}{2L}\int_L^L |f(x)|^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right). \tag{6.22}$$

Note that the identity is not true for the integration in any interval but only for $(-L, L)$ whether the functions on the right hand side is orthogonal!

Example 6.16. *If* $f(x) = \sum_{n=0}^{\infty} \frac{\cos n\pi}{2^n}$, *find the value of* $\int_\pi^\pi f^2 dx$.

Solution: In this example, $L = \pi$, $a_0 = 1$, $a_n = \frac{1}{2^n}$, $b_n = 0$, thus we have

$$\frac{1}{2\pi}\int_L^L |f(x)|^2 dx = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^{2n}} = 1 + \frac{1}{2}\left(\frac{1/4}{1-1/4}\right) = \frac{7}{6}$$

$$\int_L^L |f(x)|^2 dx = \frac{7\pi}{3}.$$

From Parseval's identity, we can get many identities of series like the one above $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^{2n}}$.

Now we discuss the calculus of Fourier series, which often deals with the limits, the differentiation, and integration of Fourier series. The tool is to use the partial sum of a series. We want to know whether the following is true.

$$\left(\lim_{x\to x_0};\ \frac{d}{dx};\ \int_\alpha^\beta dx\right) f(x) \stackrel{?}{=} \sum_{n=0}^{\infty}\left(\lim_{x\to x_0};\ \frac{d}{dx};\ \int_\alpha^\beta dx\right)\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

The partial sum forms a sequence $\{S_0(x), S_1(x), S_2(x), \cdots, S_N(x), \cdots\}$ or $\{S_N(x)\}$. Note that $S_N(x)$ has two parameters, x and N. We will discuss two kinds of convergence, a point-wise and uniform convergence in an interval. We will discuss more general sequence $f_n(x)$.

A pointwise convergence of $f_n(x)$ is defined for a fixed point x in an interval (a, b) such that $\lim_{n\to\infty} f_n(x) = f(x)$. If the partial sum $S_N(x)$, the point-wise convergence is the same as the convergence of the series.

Example 6.17. *Are the following sequences convergent? (a), $f_n(x) = \dfrac{\sin nx}{n}$; (b), $g_n(x) = nxe^{-nx+1}$.*

Solution: (a), $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{\sin nx}{n} = 0$ for any x. (b), we can use the l'Hospital's rule to get the limit. $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \dfrac{nxe}{e^{nx}} = \lim_{n\to\infty} \dfrac{xe}{xe^{nx}} = 0$ for any $x \neq 0$.

In above examples, both $f_n(x)$ and $g_n(x)$ are convergent to zero in any interval. The function $f_n(x)$ gets smaller and smaller as n gets large, while there are always points x near zero such that $g_n(x) \sim 1$ no matter how large n can be. Such an $f_n(x)$ is also called uniformly convergent, while $g_n(x)$ is not uniformly convergent in the interval $(0, \pi)$, for an example, but is uniformly convergent in any interval (a, b) if $a > 0$.

Definition 6.3. *Let $f_n(x)$ be a sequence defined in an interval $[a, b]$ and has pointwise convergence $\lim_{n\to\infty} f_n(x) = f(x)$ for any x in $[a, b]$. Given any number $\epsilon > 0$ (no matter how small it may be), if there is an integer N such that*

$$|f_n(x) - f(x)| < \epsilon \quad \textit{for any } n > N \textit{ and } x \textit{ in } [a, b], \tag{6.23}$$

then $f_n(x)$ is called uniformly convergent to $f(x)$ in $[a, b]$.

In the previous example, given an $\epsilon > 0$, for $f_n(x)\frac{\sin nx}{n}$, we have

$$|f_n(x)| = \left|\frac{\sin nx}{n}\right| \leq \left|\frac{1}{n}\right| < \epsilon$$

as long as $n \geq int(1/\epsilon) + 1$, thus we can take $N = int(1/\epsilon) + 1$. However, for $g_n(x) = nxe^{-nx+1}$, no matter how large n is, we can find an $x = 1/n$ for which $g_n(x) = 1$ which can no be smaller that arbitrary small ϵ. Thus, $g_n(x)$ is not uniformly convergent.

Definition 6.4. *For a given series $\sum_{n=0}^{\infty} u_n(x)$, if the partial sum $\{S_N(x)\}$ is uniformly convergent in an interval $[a, b]$, then the series is also uniformly convergent in the interval $[a, b]$.*

How do we know if a series is uniformly convergent without using the partial sum and the definition? The idea is to use a comparison convergent series that does have x in the series, hence it has to be uniformly convergent. This is summarized in the Weierstrass M-test theorem.

Theorem 6.5. *Weierstrass M-test theorem. Given a series* $\sum_{n=0}^{\infty} u_n(x)$ *that satisfies the following conditions*

$$(i): \quad |u_n(x)| \leq M_n \quad \textit{independent of } x \textit{ in an interval } [a,b], \tag{6.24}$$

$$(ii): \quad \sum_{n=0}^{\infty} M_n < \infty \quad \textit{the series without } x \textit{ converges}, \tag{6.25}$$

then the series is uniformly convergent in the interval $[a,b]$.

Chapter 7

Series solutions of PDEs

In this chapter, we discuss the method of separation variables for various BVP and/or IVP of PDEs. We can see the relations of the solution with the Sturm-Liouville eigenvalue problems, orthogonal expansions, and various Fourier series.

7.1 One-dimensional wave equations

We already know that we can get a series solution to the IVP-BVP problem for the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,$$
$$u(0,t) = 0, \qquad u(L,t) = 0,$$
$$u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L,$$

using

$$u(x,t) = \sum_{n=0}^{\infty} \sin \frac{n\pi x}{L} \left(b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L} \right)$$

where the coefficients are determined by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx,$$
$$b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx.$$

Note that coefficients b_n's are obtained from the half-range sine expansion of $f(x)$ and b_n^*'s are obtained from the half-range sine expansion of $g(x)$ by a constant that depends on n.

Example 7.1. *An example with non-homogeneous boundary condition.*

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \\
&u(0,t) = u_0, \qquad u(L,t) = u_0, \\
&u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L.
\end{aligned}
$$

In this case, we can use the transformation

$$
v(x,t) = u(x,t) - u_0
$$

to get the homogenous BC for $v(x,t)$

$$
\begin{aligned}
&\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < L, \\
&v(0,t) = 0, \qquad v(L,t) = 0, \\
&v(0,t) = f(x) - u_0, \qquad \frac{\partial v}{\partial t}(x,0) = g(x), \qquad 0 < x < L.
\end{aligned}
$$

The solution then will be

$$
u(x,t) = u_0 + \sum_{n=0}^{\infty} \sin\frac{n\pi x}{L}\left(b_n \cos\frac{cn\pi t}{L} + b_n^* \sin\frac{cn\pi t}{L}\right)
$$

where the coefficients are determined by

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx, \\
b_n^* &= \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi x}{L} dx.
\end{aligned}
$$

Example 7.2. *An example with a Neumann boundary condition.*

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \\
&u(0,t) = u_0, \qquad \frac{\partial u}{\partial x}(L,t) = u_0, \\
&u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L,
\end{aligned}
$$

We will get a different S-L eigenvalue problem and a different expansion.

Step 1: Let $u(x,t) = T(t)X(x)$ and plug its partial derivatives to the original PDE so that we can separate variables. The homogeneous boundary conditions require $X(0) = X'(L) = 0$. Differentiating $u(x,t) = T(t)X(x)$ with respect to t and x, respectively, we get

$$\frac{\partial u}{\partial t} = T'(t)X(x), \quad \frac{\partial^2 u}{\partial t^2} = T''(t)X(x); \qquad \frac{\partial u}{\partial x} = T(t)X'(x), \quad \frac{\partial^2 u}{\partial x^2} = T(t)X''(x).$$

The wave equation can be re-written as

$$T''(t)X(x) = c^2 T(t)X''(x), \Longrightarrow \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{7.1}$$

This is because in the last equality, the left hand is a function of t while the right hand side is a function of x, which is possible only both of them are a constant independent of t and x. We get an eigenvalues either for $X(x)$ or $T(t)$. Since we know the boundary condition for $X(x)$, naturally we should solve

$$\frac{X''(x)}{X(x)} = -\lambda \quad \text{or} \quad X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X'(L) = 0 \tag{7.2}$$

first.

Step 2: Solve the eigenvalue problem. From the Sturm-Liouville eigenvalue theory, we know that $\lambda > 0$, thus the solution is

$$X''(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

From the boundary condition $X(0) = 0$, we get $C_1 = 0$. From the boundary condition $X'(L) = 0$, we get

$$C_2 \sin\sqrt{\lambda}L = 0, \quad \Longrightarrow \quad \sqrt{\lambda}L = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, \cdots,$$

since $C_2 \neq 0$. The eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, \quad X_n(x) = \sin\frac{(2n+1)\pi}{2L}, \quad n = 0, 1, 2, \cdots, .$$

Now we solve for $T(t)$ using

$$T''(t) + c^2\lambda_n T(t) = 0. \tag{7.3}$$

The solution is (not an eigenvalue problem anymore since we have already known λ_n)

$$T_n(t) = b_n \cos\frac{(2n+1)\pi ct}{2L} + b_n^* \sin\frac{(2n+1)\pi ct}{2L}.$$

Put $X_n(x)$ and $T_n(t)$ together, we get a normal mode solution

$$u_n(x,t) = \sin\frac{(2n+1)\pi x}{2L}\left(b_n \cos\frac{(2n+1)\pi ct}{2L} + b_n^* \sin\frac{(2n+1)\pi ct}{2L}\right), \quad (7.4)$$

which satisfy the PDE, the boundary conditions, but not the initial conditions.

Step 3: Put all the normal solution together to get the series solution. The coefficients are obtained from the orthogonal expansion of the initial conditions.

The solution to the IVP-BVP of the 1D wave equation can be written as

$$u(x,t) = \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi x}{2L}\left(b_n \cos\frac{(2n+1)\pi ct}{2L} + b_n^* \sin\frac{(2n+1)\pi ct}{2L}\right) \quad (7.5)$$

which satisfies the PDE and the boundary conditions. The coefficients of b_n and b_n^* are determined from the initial conditions $u(x,0)$ and $u_t(x,0)$.

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin\frac{(2n+1)\pi x}{2L}, \qquad \Longrightarrow \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n+1)\pi x}{2L}dx,$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi x}{2L}\left(-b_n\frac{(2n+1)\pi c}{2L}\sin\frac{(2n+1)\pi ct}{2L} + b_n^*\frac{(2n+1)\pi c}{2L}\cos\frac{(2n+1)\pi ct}{2L}\right),$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi x}{2L} b_n^* \frac{(2n+1)\pi c}{2L}, \qquad \Longrightarrow$$

$$b_n^* = \frac{\dfrac{2L}{(2n+1)\pi c}\displaystyle\int_0^L g(x)\sin\frac{(2n+1)\pi x}{2L}dx}{\displaystyle\int_0^L \left(\sin\frac{(2n+1)\pi x}{2L}\right)^2 dx} = \frac{4}{(2n+1)\pi c}\int_0^L g(x)\sin\frac{(2n+1)\pi x}{2L}dx.$$

Example 7.3. *An example of a different PDE with a lower order term.*

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + a^2 u = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\\
&u(0,t) = 0, \qquad u(L,t) = 0,\\
&u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).
\end{aligned}$$

Solution: The method of separation variables with $u(x,t) = T(t)X(x)$ will lead to

$$\frac{T''(t) + a^2 T(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \quad (7.6)$$

We still have $\lambda_n = (\frac{n\pi}{L})^2$ and $X_n(x) = \sin\frac{n\pi x}{L}$. But the solution of $T_n(t)$ will be different.

$$T_n(t) = b_n \cos t\sqrt{a^2 + \frac{c^2n^2\pi^2}{L^2}} + b_n^* \sin t\sqrt{a^2 + \frac{c^2n^2\pi^2}{L^2}}. \tag{7.7}$$

The solution then is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(b_n \cos t\sqrt{a^2 + \frac{c^2n^2\pi^2}{L^2}} + b_n^* \sin t\sqrt{a^2 + \frac{c^2n^2\pi^2}{L^2}}\right) \tag{7.8}$$

with b_n being the coefficient of the half range sine expansion of $f(x)$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \tag{7.9}$$

and

$$b_n^* = \frac{2}{L\alpha_n}\int_0^L g(x)\sin\frac{n\pi x}{L}dx, \quad \text{where} \quad \alpha_n = \sqrt{a^2 + \frac{c^2n^2\pi^2}{L^2}}. \tag{7.10}$$

7.2 One-dimensional heat equations

A one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2},$$

is a good mathematical model of the temperature distribution in a rod in which c is called the heat conductivity. We can check that

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-\dfrac{x^2}{4c^2t}} \tag{7.11}$$

is a solution to the PDE. It is called a fundamental solution of the heat equation. The solution to the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \\ &u(x,0) = f(x), \end{aligned}$$

is

$$u(x,t) = \int_{-\infty}^{\infty} \frac{f(\xi)}{\sqrt{4c^2\pi t}}\, e^{-\dfrac{(x-\xi)^2}{4c^2t}}\, d\xi. \tag{7.12}$$

Now we consider the boundary and initial value problems

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,$$
$$u(0,t) = 0, \qquad u(L,t) = 0,$$
$$u(x,0) = f(x),$$

using the method of separation of variables.

Step 1: Let $u(x,t) = T(t)X(x)$ and plug its partial derivatives to the original PDE so that we can separate variables. The homogeneous boundary conditions require $X(0) = X(L) = 0$. Differentiating with $u(x,t) = T(t)X(x)$ with t and x respectively, we get

$$\frac{\partial u}{\partial t} = T'(t)X(x); \qquad \frac{\partial u}{\partial x} = T(t)X'(x), \quad \frac{\partial^2 u}{\partial x^2} = T(t)X''(x).$$

The heat equation can be re-written as

$$T'(t)X(x) = c^2 T(t)X''(x), \Longrightarrow \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{7.13}$$

This is because in the last equality, the left hand is a function of t while the right hand side is a function of x, which is possible only both of them are a constant independent of t and x. We get an eigenvalues either for $X(x)$ or $T(t)$. Since we know the boundary condition for $X(x)$, naturally we should solve

$$\frac{X''(x)}{X(x)} = \lambda \quad \text{or} \quad X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0 \tag{7.14}$$

first.

Step 2: Solve the eigenvalue problem. From the Sturm-Liouville eigenvalue theory, we know that $\lambda > 0$, thus the solution is

$$X''(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

From the boundary condition $X(0) = 0$, we get $C_1 = 0$. From the boundary condition $X(L) = 0$, we get

$$C_2 \sin\sqrt{\lambda}L = 0, \quad \Longrightarrow \quad \sqrt{\lambda}L = n\pi, \quad n = 1, 2, \cdots,$$

since $C_2 \neq 0$. The eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \cdots, .$$

Now we solve for $T(t)$ using

$$T'(t) + c^2 \lambda_n T(t) = 0. \tag{7.15}$$

The solution is (not an eigenvalue problem anymore since we have already known λ_n)

$$T_n(t) = b_n e^{c^2 \lambda_n t} = b_n e^{-c^2 (\frac{n\pi}{L})^2 t}.$$

Put $X_n(x)$ and $T_n(t)$ together, we get a normal mode solution

$$u_n(x,t) = \sin \frac{n\pi x}{L} b_n e^{-c^2 (\frac{n\pi}{L})^2 t}, \tag{7.16}$$

which satisfy the PDE, the boundary conditions, but not the initial conditions.

Step 3: Put all the normal solution together to get the series solution. The coefficients are obtained from the orthogonal expansion of the initial conditions.

The solution to the IVP-BVP of the 1D wave equation can be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-c^2 (\frac{n\pi}{L})^2 t} \tag{7.17}$$

which satisfies the PDE and the boundary conditions, but the initial condition. The coefficients of b_n are determined from the initial conditions $u(x,0)$,

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad \Longrightarrow \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx,$$

which again is the half range sine expansion of the initial condition $u(x,0) = f(x)$.

Example 7.4. *Solve the following heat equation*

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi,$$
$$u(0,t) = 0, \qquad u(\pi,t) = 0,$$
$$u(0,t) = 100, \qquad 0 < x < \pi.$$

Solution: Since the boundary condition is the homogeneous Dirichlet boundary condition ($u(0,t) = u(L,t) = 0$) at two ends, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx e^{-c^2 n^2 t}$$

with

$$b_n = \frac{2}{\pi}\int_0^L 100\sin nx dx = \frac{200}{\pi}\left(\frac{\cos n\pi}{n}\Big|_0^\pi\right) == \frac{200}{n\pi}\left(1-\cos n\pi\right).$$

The solution then is

$$u(x,t) = \sum_{n=0}^{\infty} \frac{400}{(2n+1)\pi}\sin(2n+1)x\, e^{-c^2(2n+1)^2 t}.$$

Example 7.5. *Solve the following heat equation*

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0<x<L,$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad u(L,t) = 0,$$
$$u(0,t) = 100, \qquad 0<x<L.$$

Solution: Note that the different boundary condition of the solution at $x = 0$. The normal modes solution still can be written as $u_n(x,t) = T_n(t)X_n(x)$ in which $X_n(x)$ and $T_n(x)$ satisfy the following equations

$$\frac{X_n''}{X_n} = -\lambda_n, \qquad \frac{T_n''}{c^2 T_n} = -\lambda_n.$$

We need to solve a different Sturm-Liouville eigenvalue problem

$$\frac{X_n''}{X_n} = -\lambda_n, \qquad X'(0) = 0, \quad X(L) = 0.$$

Once again we have

$$X(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x, \quad X'(x) = \sqrt{\lambda}\left(-C_1\sin\sqrt{\lambda}x + C_2\cos\sqrt{\lambda}x\right).$$

The boundary condition $X'(0) = 0$ leads to $C_2 = 0$. The other boundary condition $X(L) = 0$ leads to

$$X(L) = C_1\cos\sqrt{\lambda}L = 0, \quad \Longrightarrow \quad \sqrt{\lambda}L = \frac{\pi}{2} + n\pi,\ n = 0, 1, \cdots$$

We get $\lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$, $X_n = \cos\frac{(2n+1)\pi}{2L}x$. The solution for $T(t)$, $T'(t) + \lambda_n c^2 T(t) = 0$ is

$$T_n(t) = e^{-c^2\left((\frac{1}{2}+n)\frac{\pi}{L}\right)^2 t}. \tag{7.18}$$

The solution to the initial-boundary value problem is

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-c^2\left((\frac{1}{2}+n)\frac{\pi}{L}\right)^2 t} \cos\frac{(2n+1)\pi}{2L}\, x. \tag{7.19}$$

The coefficient c_n is determined from the initial condition

$$u(x,0) = \sum_{n=0}^{\infty} c_n \cos\frac{(2n+1)\pi}{2L}\, x = f(x),$$

which gives

$$c_n = \frac{\displaystyle\int_0^L f(x)\cos\frac{(2n+1)\pi}{2L}\, x\, dx}{\displaystyle\int_0^L \cos^2\frac{(2n+1)\pi}{2L}\, x dx} = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n+1)\pi}{2L}\, x\, dx. \tag{7.20}$$

7.2.1 The steady state solution of 1D heat equations

A steady state solution to an ODE/PDE is a function independent of time t that satisfies the following.

- A solution to the ODE/PDE.
- Satisfies the boundary condition but has nothing to do with the initial condition.
- It is independent of time t, *i.e.*, $\frac{\partial u}{\partial t} = 0$.

The steady state solution is the result of long time behavior. Note that, not all the problems have a steady state solution.

Example 7.6. *Find the steady state solution*

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,$$
$$u(0,t) = T_1, \qquad u(L,t) = T_2,$$
$$u(0,t) = f(x), \qquad 0 < x < L.$$

Solution: The steady state solution is the solution to the following problem $u(x,t) \Longrightarrow u_1(x)$,

$$0 = c^2\frac{d^2 u_1}{dx^2}, \qquad 0 < x < L,$$
$$u_1(0) = T_1, \qquad u_1(L) = T_2.$$

The general solution is $u_1(x) = C_1 + C_2 x$. The BC $u_1(0) = T_1$ leads to $C_1 = T_1$ and $u_1(L) = T_2$ leads to the steady state solution

$$u_1(x) = T_1 + \frac{U_2 - U_1}{L} x.$$

One application of the steady state solution is to transform a non-homogeneous boundary condition to a homogeneous one. If we want to solve the PDE above for anytime (not just long term behavior), then we can define $w(x,t) = u(x,t) - u_1(x)$ and $w(x)$ will satisfies the homogeneous boundary condition and is the solution to the following

$$\begin{aligned} &\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2}, \qquad 0 < x < L, \\ &w(0,t) = 0, \qquad w(L,t) = 0, \\ &w(0,t) = f(x) - u_1(x), \qquad 0 < x < L. \end{aligned}$$

Once we have solved $w(x,t)$, we get back the solution $u(x,t) = w(x,t) + u_1(x)$.

Note that not all the time dependent problems have steady state solutions. For example, for the heat equation with $u(0,t) = \sin t$, the boundary condition depends on t and hence there is no steady state solutions.

7.3 Two-dimensional Laplace equations

In this section, we consider the series solution to Laplace or Poisson equations on rectangular domain. A Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{or} \quad u_{xx} + u_{yy} = 0, \qquad (x,y) \in R, \tag{7.21}$$

$$u(x,y)\Big|_R = w(x,y), \quad \text{or} \quad \frac{\partial u}{\partial n}(x,y)\Big|_R = g(x,y), \tag{7.22}$$

or other boundary conditions, where $\frac{\partial u}{\partial n}(x,y)$ is the directional derivative of $u(x,y)$ along the outer normal directions (n, $|n| = 1$).

In engineering, we can use the gradient operator to represent the Laplace/Poisson equation in any dimensions using $\nabla^2 u = 0$, or $\Delta u = 0$, for example, in two dimensions, we have

$$\nabla^2 u = \nabla \cdot \nabla u = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{7.23}$$

The operator $\nabla^2 = \Delta$ is called the Laplace operator. Note that the solution of a Laplace equation can be considered as the steady state solution of the 2D heat

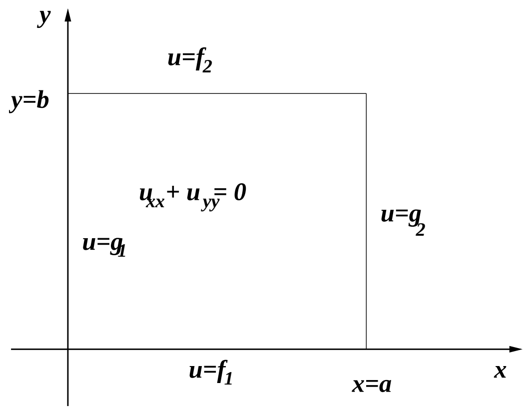

Figure 7.1. *A diagram of a Laplace equation defined on a rectangular domain with a Dirichlet boundary condition.*

equation (or a wave equation)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} 0, \qquad (x,y) \in R, \tag{7.24}$$

$$u(x,y)\Big|_R = w(x,y), \quad \text{or} \quad \frac{\partial u}{\partial n}(x,y)\Big|_R = g(x,y), \tag{7.25}$$

$$u(x,y,0) = f(x,y) \tag{7.26}$$

for an arbitrary two dimensional function $f(x,y)$. Applications of Laplace equations can be found in potential flows, ideal flows, electro-magnetics. A conservative vector field, $div(\mathbf{u}) = curl \times \mathbf{u} = 0$ can be represented as a potential of a scale function, $\mathbf{u} = \nabla\varphi$ and $\Delta\varphi = 0$, for an example, the gravity field. We will have only one scalar equation instead of three equations of a vector field.

It is easy to check that $u(x,y) = \frac{1}{2\pi} \log \sqrt{x^2+y^2}$ is a solution to the 2D Laplace equation. It is called the fundamental solution, which corresponds to a point source (charge). In three dimension, the fundamental solution is $u(x,y,z) = \frac{1}{x^2+y^2+z^2}$. The fundamental solution satisfies the PDE but not the boundary conditions in general.

To use the method os separation variables, we wish to have at least two homogeneous boundary conditions. Since the problem is linear, we can split the problem into four sub-problems, see Figure 7.2 for an illustration. The final solution will be the sum of the solutions of the sub-problems.

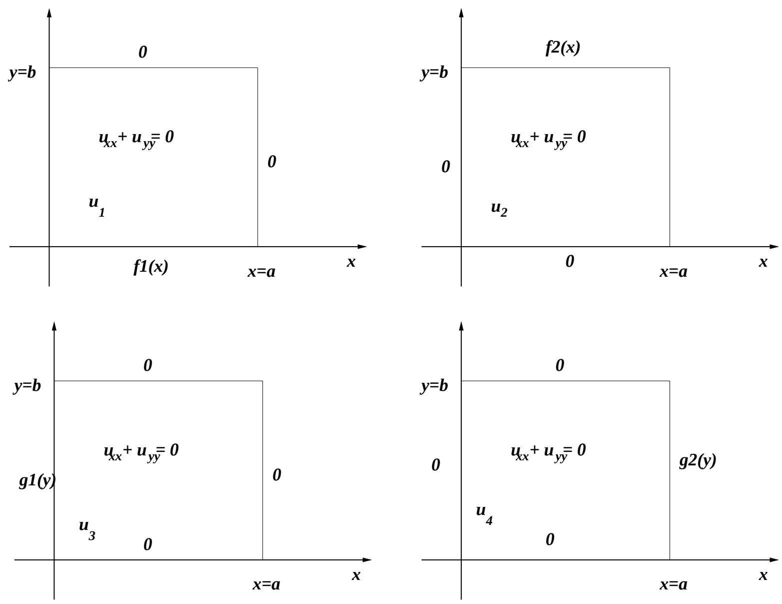

Figure 7.2. *A diagram that we can decomposition of the solution of a Laplace equation on a rectangular domain into four sub-problems.*

We solve one of the problems in Figure 7.2

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x, y) \in R, \tag{7.27}$$

$$u(x, b) = f_2(x), \qquad u(x, 0) = 0, \qquad u(0, y) = 0, \qquad u(a, y) = 0. \tag{7.28}$$

We set $u(x, y) = X(x)Y(y)$ and get the separate Sturm-Liouville eigenvalue problem

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \tag{7.29}$$

with $X(0) = 0$ and $X(a) = 0$. We can solve for $X(x)$ first to get

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \qquad X_n(x) = \sin\frac{n\pi x}{a}, \quad n = 1, 2, \cdots . \tag{7.30}$$

Now we solve $Y(y)$ from $-\frac{Y''}{Y} = \lambda = (\frac{n\pi}{a})^2$. The solution can be expressed by

$$Y_n(y) = b_n e^{-\frac{n\pi x}{a}} + b_n^* e^{\frac{n\pi x}{a}} \tag{7.31}$$

or the hyperbolic sine and cosine functions

$$Y_n(y) = B_n \sinh\frac{n\pi y}{a} + B_n^* \cosh\frac{n\pi y}{a}. \tag{7.32}$$

The hyperbolic sine and cosine functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh = \frac{e^x + e^{-x}}{2}, \tag{7.33}$$

respectively. They are linear independent whose Wronskian

$$det \begin{pmatrix} \sinh x & \cosh x \\ \cosh x & \sinh x \end{pmatrix} = -1 \neq 0 \quad \text{for any } x. \tag{7.34}$$

The hyperbolic cosine and sine functions have some similar properties as sine and cosine functions such as $\sinh(0) = 0$, $\cosh(0) = 1$. From $Y(0) = 0$, we get $B_n^* = 0$ and we can write the solution to the Laplace equation as

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

Plug the non-homogenous boundary condition along $y = b$, we get

$$u(x, b) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a} = f_2(x),$$

we get the coefficients

$$B_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f_2(x) \sin \frac{n\pi x}{a} dx,$$
$$B_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \sin \frac{n\pi x}{a} dx.$$

Example 7.7. *Solve the Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x, y < 1,$$
$$u(x, 1) = x(1 - x), \qquad u(x, y) = 0, \quad \textit{on other three boundaries.}$$

Solution: In this example, $a = 1$, $b = 1$, we have

$$\begin{aligned} B_n &= \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_2(x) \sin \frac{n\pi x}{a} dx = 2 \int_0^1 x(1-x) \sin n\pi x dx \\ &= \frac{2}{n\pi}(-\cos n\pi x) x(1-x) \Big|_0^1 + \frac{2}{n\pi} \int_0^1 (1-2x) \cos n\pi x dx \\ &= \frac{2}{(n\pi)^2}(1-2x) \sin n\pi x \Big|_0^1 - \frac{2}{(n\pi)^2} \int_0^1 (-2) \sin n\pi x dx \\ &= \frac{4}{(n\pi)^3}(-\cos n\pi x) \Big|_0^1 = -\frac{4}{(n\pi)^3} (\cos n\pi - 1) \\ &= \frac{8}{(2k-1)^3 \pi^3}, \qquad k = 1, 2, \cdots . \end{aligned}$$

The solution then is, see also Figure 7.3 and the Maple file Laplace.mws,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3 \pi^3} \frac{\sin(2n-1)\pi x \, \sinh((2n-1)\pi y}{\sinh(2n-1)\pi}$$

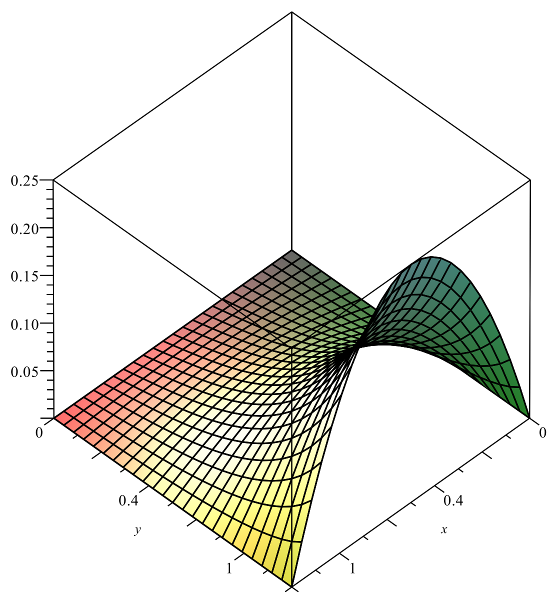

Figure 7.3. *Plot of the partial sum $S_{20}(x, y)$ of the series solution.*

How do we find the solution for the first case, *i.e.,* $u_1(x, 0) = f_1(x)$ and $u_1(x, y) = 0$ on other three boundaries? We can repeat the method of separation of variables; or we can change the problem to a one that we have already solved.

Let $\bar{y} = b - y$, $\bar{x} = x$, then

$$u_1(x, y) = u_1(\bar{x}, b - \bar{y}) \overset{define}{=} \bar{u}_1(\bar{x}, \bar{y})$$

. We have the following

$$\frac{\partial^2 \bar{u}_1}{\partial \bar{x}^2} = \frac{\partial^2 u_1}{\partial x^2}, \qquad \frac{\partial \bar{u}_1}{\partial \bar{y}} = -\frac{\partial u_1}{\partial y}, \qquad \frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2} = \frac{\partial^2 u_1}{\partial y^2}$$

$$\frac{\partial^2 \bar{u}_1}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}_1}{\partial \bar{y}^2} = \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0,$$

$$\bar{u}_1(0, \bar{y}) = u_1(0, b - y) = 0, \quad \bar{u}_1(a, \bar{y}) = u_1(a, b - y) = 0, \quad \bar{u}_1(\bar{x}, 0) = u_1(x, b) = 0,$$

$$\bar{u}_1(\bar{x}, b) = u_1(x, 0) = f_1(x).$$

We apply the previous solution formula to get

$$\bar{u}_1(\bar{x}, \bar{y}) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi\bar{x}}{a} \sinh\frac{n\pi\bar{y}}{a}, \qquad A_n = \frac{2}{a \sinh\frac{n\pi b}{a}} \int_0^a f_1(x) \sin\frac{n\pi x}{a} dx.$$

We switch to the original coordinates to get

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi(b - y)}{a}.$$

For the non-homogeneous part along the boundaries $x = 0$ and $x = a$, we can use the symmetry arguments by switching x with y, and a with b, we can get a formula for the entire problem

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi(b - y)}{a} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} + \\ &= \sum_{n=1}^{\infty} C_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi(a - x)}{b} + \sum_{n=1}^{\infty} D_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi x}{b}, \end{aligned} \tag{7.35}$$

where the coefficients are

$$A_n = \frac{2}{a \sinh\frac{n\pi b}{a}} \int_0^a f_1(x) \sin\frac{n\pi x}{a} dx, \quad B_n = \frac{2}{a \sinh\frac{n\pi b}{a}} \int_0^a f_2(x) \sin\frac{n\pi x}{a} dx.$$

$$C_n = \frac{2}{a \sinh\frac{n\pi a}{b}} \int_0^b g_1(y) \sin\frac{n\pi y}{b} dx, \quad D_n = \frac{2}{b \sinh\frac{n\pi a}{b}} \int_0^b g_2(x) \sin\frac{n\pi y}{b} dx.$$

7.4 Method of Separation variables in Polar coordinates

In many applications, it is preferred to use polar coordinates especially when we deal with circles, annulus etc. Often we can solve two or three dimensional problem using one dimensional settings if the problem posses axis-symmetry. How will PDEs be changed using the polar/cylindrical coordinates? We know that

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2}, \quad \theta = \arctan(y/x). \tag{7.36}$$

For a function $u(x, y)$, we can represent the function and its partial derivatives using (r, θ)

$$u(x, y) = u(r\cos\theta, r\sin\theta) = \bar{u}(r, \theta). \tag{7.37}$$

For simplicity, we often omit the bar if there is no confusion occurring. Next, we replace the partial derivative as well using the chair rule. We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

From $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$, we also have

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \frac{x}{r},$$
$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(\frac{y}{x})^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{r\sin\theta}{r^2}$$

Thus we get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\left(-\frac{r\sin\theta}{r^2}\right) = \frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial \theta}\frac{y}{r^2}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{y}{r} + \frac{\partial u}{\partial \theta}\frac{x}{r^2} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r},$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial \theta}\frac{y}{r^2}\right)\cdots$$

The derivations are long and tedious. Fortunately, for most practical PDES, vector relations, we can find the conversions through mathematical handbooks or online tools. The Laplace equation in polar coordinates in two-dimensions is

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{7.38}$$

Note that there is no $\frac{\partial u}{\partial \theta}$ term in the expression above. We can use the dimension analysis to figure out the coefficient in the above terms knowing that θ is a dimensionless quantity. For the radial symmetric case, that is, the solution is independent of θ, we have the simplified Laplace equation

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0. \tag{7.39}$$

Series solution to the Laplace equation in circular regions

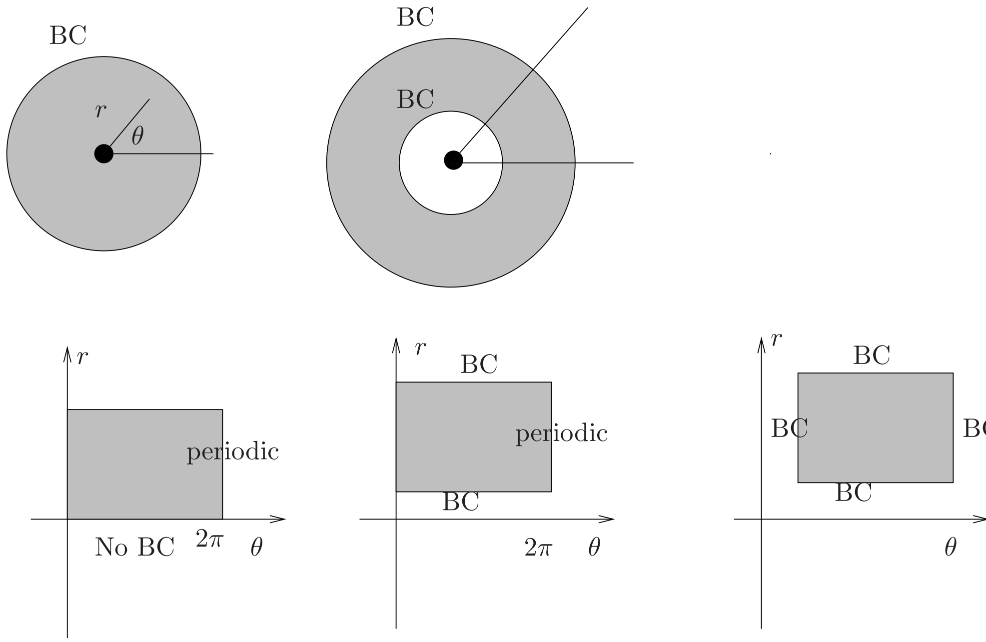

Figure 7.4. *Diagrams of domains and boundary conditions that may be better solved in polar coordinates.*

Consider the Laplace equation on a circular region $r \le a$,

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r \le a,\ 0 \le \theta \le 2\pi, \tag{7.40}$$

with a Dirichlet boundary condition at $r = a$, $u(a, \theta) = f(\theta)$. Note that $r = 0$ is an interior point, not a boundary. There is no boundary condition at $r = 0$ except that the solution should be bounded. With the method of separation of variables, we set $u(r, \theta) = R(r)\Theta(\theta)$, the PDE becomes

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0, \tag{7.41}$$

$$\frac{R'' + \frac{1}{r}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0, \tag{7.42}$$

$$-r^2\left(\frac{R'' + \frac{1}{r}R'}{R}\right) = \frac{\Theta''}{\Theta} = \lambda. \tag{7.43}$$

We have two related Sturm-Liouville eigenvalue problems

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta(0) = \Theta(2\pi), \tag{7.44}$$

$$r^2R'' + rR' + \lambda R, \qquad 0 < r < a. \tag{7.45}$$

The boundary condition $\Theta(0) = \Theta(2\pi)$ is called a periodic one. We do not know the solution of $R(0)$ and $R(a)$ except that they are bounded. Thus, we should solve the first Sturm-Liouville eigenvalue problem first. If $\lambda < 0$, we would have

$$\Theta(\theta) = C_1 e^{-\sqrt{\lambda}\theta} + C_2 e^{\sqrt{\lambda}\theta}$$

which can not be peridot, neither the case $\lambda = 0$ for which we have $\Theta(\theta) = C_1 + C_2\theta$. Thus we must have to have $\lambda > 0$ for which the solution is

$$\Theta(\theta) = C_1 \cos\sqrt{\lambda}\theta + C_2 \sin\sqrt{\lambda}\theta.$$

Apply the periodic boundary condition, we should have

$$\Theta(\theta) = C_1 \cos\sqrt{\lambda}(\theta + 2\pi) + C_2 \sin\sqrt{\lambda}(\theta + 2\pi) = C_1 \cos\sqrt{\lambda}\theta + C_2 \sin\sqrt{\lambda}\theta,$$

which leads to $2\pi\sqrt{\lambda} = 2\pi n$, $n = 0, 1, 2, \cdots$, or $\lambda^2 = n^2$. Note that in this case, $n = 0$ is a valid solution. The eigenfunction then is

$$\Theta(\theta) = a_n \cos n\theta + b_n \sin n\theta, \qquad n = 0, 1, 2, \cdots, . \tag{7.46}$$

Now we use the second ODE to determine the coefficients

$$r^2R'' + rR' - n^2R = 0, \qquad 0 < r < a. \tag{7.47}$$

It is an Euler's equation, see Appendix **??**. The indicial equation is

$$\alpha(\alpha - 1) + \alpha - n^2 = 0 \tag{7.48}$$

whose solutions are

$$R(r) = C_n \left(\frac{r}{a}\right)^n + \bar{C}_n \left(\frac{r}{a}\right)^{-n}, \qquad n = 1, 2, \cdots, \tag{7.49}$$

Since the solution is bounded at $r = 0$, we have to have $\bar{C}_n = 0$, so the series solution is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta). \tag{7.50}$$

We apply the boundary condition to get

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = f(\theta), \tag{7.51}$$

which is the Fourier series expansion of $f(\theta)$ with

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta dx,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta, \qquad n = 1, 2, \cdots.$$

Example 7.8. *Find the steady state solution of the following*

$$\frac{\partial u}{\partial t} = \Delta u, \qquad x^2 + y^2 < 1,$$
$$u(1,\theta,t) = 100 - e^{-t}, \qquad u(r,\theta,0) = r\sin\theta.$$

Solution: The steady state solution of the following

$$\Delta u = 0, \qquad x^2 + y^2 < 1,$$
$$u(1,\theta) = 100.$$

We can compute the coefficients of the Fourier series of $f(\theta) = 100$ to get $a_0 = 100$, $a_n = 0$ and b_n=0. The Fourier series is itself corresponding to $\cos 0x$ and the solution is $u\theta) = 100$!

How about $u(1,\theta,t) = \sin 5\theta + \cos 7\theta$? The steady state solution is the normal modes solution $u(\theta) = r^5 \sin 5\theta + r^7 \cos 7\theta$.

How about $u(1,\theta,t) = 100$ if $0 < \theta < \pi$ and $u(1,\theta,t) = 0$ if $\pi < \theta < 2\pi$? We have

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} 100 d\theta = 50, \qquad a_n = \frac{1}{\pi}\int_0^{\pi} 100\cos n\theta d\theta = 0,$$
$$b_n = \frac{1}{\pi}\int_0^{\pi} 100\sin n\theta d\theta = \frac{100}{\pi} - \frac{\cos n\theta}{n}\bigg|_0^{\pi} = \frac{100}{n\pi}\left(1 - (-1)^n\right).$$

The solution is

$$u(r,\theta) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-(-1)^n\right) r^n \sin n\theta,$$

The series is uniformly convergent if $r \le \alpha < 1$, but not in $(0, 1)$.

Chapter 8

Fourier and Laplace transform

We have seen the power of various Fourier series in solving boundary value problems of partial differential equations and their applications. If we et L goto ∞, and replace the summation with integration, then we will have the Fourier transform. The Fourier transform is very useful in terms of theoretical analysis, get analytic solutions of certain PDEs especially those defined in the entire space.

8.1 From the Fourier series to Fourier integral representation

Give a function in $L^2(-L, L)$, we have the Fourier expansion

$$f(x) = \sum_{n=0}^{\infty} \left\{ \frac{1}{L} \left(\int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} dt \right) \cos \frac{n\pi x}{L} + \frac{1}{L} \left(\int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} dt \right) \sin \frac{n\pi x}{L} \right\}.$$

Let $\frac{n\pi t}{L} = \omega$ or $\frac{1}{L} = \frac{1}{\pi}\frac{w}{n} = \frac{1}{\pi}\Delta\omega$. The expression above becomes

$$f(x) = \sum_{n=0}^{\infty} \left\{ \frac{1}{\pi} \left(\int_{-L}^{L} f(t) \cos \omega t dt \right) \cos \omega x + \frac{1}{\pi} \left(\int_{-L}^{L} f(t) \sin \omega t dt \right) \sin \omega x \right\}.$$

As $p \to \infty$, we get

$$f(x) = \int_0^{\infty} \left(A(\omega) \cos \omega x + B(\omega) \sin \omega x \right) d\omega, \tag{8.1}$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt, \qquad \text{cosine transform of } f(x) \tag{8.2}$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt, \qquad \text{sine transform of } f(x) \tag{8.3}$$

The expression (8.1) is called Fourier integral representation of $f(x)$ which converges to $f(x)$ is $f(x)$ is continuous at a point x; and to $(f(x-)+f(x+))/2$ if $f(x)$ is piecewise continuous.

If we put cosine and sine transforms together and use the trig-identity $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$, we derive the Fourier transform of $f(x)$ below.

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\left(\cos\omega t\cos\omega x+\sin\omega t\sin\omega x\right)dt\,d\omega\\
&= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos(x-t)dt\,d\omega\\
&= \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\left(e^{i\omega(x-t)}+e^{i\omega(x+t)}\right)dt\,d\omega\\
&= \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty f(t)e^{i\omega(x-t)}dt\,d\omega-\frac{1}{2\pi}\int_{-\infty}^0\int_{-\infty}^\infty f(t)e^{i\bar{\omega}(x-t)}dt\,d\bar{\omega}\\
&= \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(t)e^{i(\omega-t)}dt\,d\omega\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)e^{-i\omega t}dt\,d\omega\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(\omega)e^{i\omega x}d\omega.
\end{aligned}$$

The expression

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-i\omega x}dx \tag{8.4}$$

is called the *Fourier transform* of $f(x)$. From the derivation above, we also have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{i\omega x}d\omega, \tag{8.5}$$

which is called the *inverse Fourier transform*.

Example 8.1. *Find the Fourier transform of $f(x)=e^{-a|x|}$, where $a>0$ is a constant.*

The problem can be solved directly from the definition.

$$
\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ax} e^{-i\omega x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ax} e^{-i\omega x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{i\omega x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ax} e^{-i\omega x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{(i\omega - a)x}}{i\omega - a} \right|_0^{\infty} + \frac{1}{\sqrt{2\pi}} \left. \frac{-e^{(i\omega + a)x}}{i\omega + a} \right|_0^{\infty} \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a + i\omega} + \frac{1}{a - i\omega} \right) \\
&= \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}.
\end{aligned}
$$

Example 8.2. *Find the Fourier transform of the square function* $f(x) = \begin{cases} 1 & \text{if} \quad |x| < a, \\ 0 & \text{if} \quad |x| > a \end{cases}$, *where* $a > 0$ *is a constant.*

The problem can be solved directly from the definition.

$$
\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\omega x} dx = -\frac{1}{\sqrt{2\pi}} \left. \frac{e^{-i\omega x}}{i\omega} \right|_{-a}^{a} \\
&= -\frac{1}{\omega i \sqrt{2\pi}} \left(e^{-i\omega a} - e^{i\omega a} \right) \\
&= -\frac{1}{\omega i \sqrt{2\pi}} \Big(\cos\omega a - i \sin\omega a - (\cos\omega a + i \sin\omega a) \Big) \\
&= \frac{2}{\sqrt{2\pi}} \frac{\sin\omega a}{\omega} = \sqrt{\frac{2}{\pi}} \frac{\sin\omega a}{\omega}.
\end{aligned}
$$

Note that the $\omega = 0$ is a removable singularity since

$$
\hat{f}(0) = \lim_{\omega \to 0} \sqrt{\frac{2}{\pi}} \frac{\sin\omega a}{\omega} = a\sqrt{\frac{2}{\pi}}.
$$

Example 8.3. *Find the Fourier transform of a point source function* $f(x) = \delta(x)$, *a special function defined only in the sense of distribution.*

$$\int f(x)\delta(x-\alpha)dx = f(\alpha) \tag{8.6}$$

if α is in the domain of integration.

The problem can be solved directly from the definition.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}}.$$

Note that the point source function $\delta(x)$ is called a Dirac delta function, which can be regarded as a 'limit' of the following non-negative function whose has a unit area

$$\delta_\epsilon(x) = \begin{cases} \dfrac{1-|x|}{\epsilon} & \text{if} \quad |x| \le \epsilon, \\ 0 & \text{Otherwise} \end{cases}. \tag{8.7}$$

It is easy to show that

$$\lim_{\epsilon\to 0} \int f(x)\delta_\epsilon(x-\alpha)dx = f(\alpha).$$

Such a $\delta_\epsilon(x)$ is unique, for example, the following function plays the same role

$$\delta_\epsilon(x) = \begin{cases} \dfrac{1}{4\epsilon}\left(1+\cos\dfrac{\pi x}{2\epsilon}\right) & \text{if} \quad |x| \le 2\epsilon, \\ 0 & \text{Otherwise} \end{cases}. \tag{8.8}$$

The Dirac delta function is the weak derivative of the Heaviside function

$$H(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{Otherwise.} \end{cases} \tag{8.9}$$

8.1.1 Properties of Fourier transform

Theorem 8.1. *Let $\hat{u}$ be the Fourier transform of a function $u \in L^2$, then*

$$\widehat{\frac{\partial u}{\partial x}} = i\omega\hat{u}, \tag{8.10}$$

$$\widehat{\frac{\partial \hat{u}}{\partial \omega}} = -ixu, \tag{8.11}$$

$$\widehat{\widehat{u}} = u. \tag{8.12}$$

Proof: From the definition, we have

$$\widehat{\hat{u}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}\, d\omega = u.$$

For the partial derivatives, first from the definition, we have

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\frac{\partial u}{\partial x}} e^{i\omega x} d\omega.$$

On the other hand, if we take the partial derivative with respect to x assuming we can switch the integration and the partial derivatives, we get

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left(e^{i\omega x} \widehat{u}(\omega) \right) d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega e^{i\omega x} \widehat{u}(\omega)\, d\omega. \end{aligned}$$

Thus the inside expressions have to be the same, that is, $\widehat{\frac{\partial u}{\partial x}} = i\omega \hat{u}$.

If we switch the position between ω and x, u and $\hat{u}$, we get

$$\frac{\partial \hat{u}}{\partial \omega} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \widehat{\frac{\partial \hat{u}}{\partial \omega}}\, dx.$$

and by differentiating the Fourier transform with respect to ω we get

$$\frac{\partial \hat{u}}{\partial \omega} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -ixe^{-i\omega x} u\, dx.$$

Thus we get $\widehat{\frac{\partial \hat{u}}{\partial \omega}} = -ixu$, which completes the proof.

Parseval's relation: Under the Fourier transform, we have $\|\hat{u}\|_2 = \|u\|_2$ or

$$\int_{-\infty}^{\infty} |\hat{u}|^2 d\omega = \int_{-\infty}^{\infty} |u|^2 dx\,. \tag{8.13}$$

It is easy to generalize the equality, to set

$$\widehat{\frac{\partial^m u}{\partial x^m}} = (i\omega)^m\, \hat{u} \tag{8.14}$$

i.e., we remove the derivatives of one variable.

8.2 Use Fourier transform to solve some of PDEs of Cauchy problems

The Fourier transform is a powerful tool to solve PDEs, as illustrated below.

Example 8.4.

Consider

$$u_t + au_x = 0\,, \qquad -\infty < x < \infty, \quad t > 0\,, \quad u(x,0) = u_0(x)$$

which is called an advection equation, or a one-way wave equation. This is a Cauchy problem since the spatial variable is defined in the entire space and $t \geq 0$. On applying the FT to the equation and the initial condition,

$$\widehat{u_t} + \widehat{au_x} = 0, \quad \text{or} \quad \widehat{u}_t + ai\omega\hat{u} = 0, \qquad \hat{u}(\omega,0) = \hat{u}_0(\omega)$$

i.e., we get an ODE

$$\hat{u}(\omega,t) = \hat{u}(\omega,0)\,e^{-ia\omega t} = \hat{u}_0(\omega)\,e^{-ia\omega t}$$

for $\hat{u}(\omega)$. The solution to the original advection equation is thus

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x}\,\hat{u}_0(\omega)\,e^{-ia\omega t}\,d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega(x-at)}\,\hat{u}_0(\omega)\,d\omega \\ &= u(x-at,0)\,, \end{aligned}$$

on taking the inverse Fourier transform. It is notable that the solution for the advection equation does not change shape, but simply propagates along the characteristic line $x - at = 0$, and that

$$\|u\|_2 = \|\hat{u}\|_2 = \|\hat{u}(\omega,0)e^{-ia\omega t}\|_2 = \|\hat{u}(\omega,0)\|_2 = \|u_0\|_2\,.$$

Example 8.5.

Consider

$$u_t = \beta u_{xx}, \qquad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = u_0(x), \quad \lim_{|x|\to\infty} u = 0\,,$$

involving the heat (or diffusion) equation. On again applying the Fourier transform to the PDE and the initial condition,

$$\widehat{u_t} = \widehat{\beta u_{xx}}, \quad \text{or} \quad \widehat{u}_t = \beta(i\omega)^2\hat{u} = -\beta\omega^2\hat{u}, \qquad \hat{u}(\omega,0) = \hat{u}_0(\omega)\,,$$

and the solution of this ODE is

$$\hat{u}(\omega,t) = \hat{u}(\omega,0)\,e^{-\beta\omega^2 t}\,.$$

Consequently, if $\beta > 0$, from the Parseval's relation, we have

$$\|u\|_2 = \|\hat{u}\|_2 = \|\hat{u}(\omega,0)e^{-\beta\omega^2 t}\|_2 \le \|u_0\|_2 \,.$$

Actually, it can be seen that $\lim_{t\to\infty} \|u\|_2 = 0$ and the second order partial derivative term is call a diffusion or dissipative. If $\beta < 0$, then $\lim_{t\to\infty} \|u\|_2 = \infty$, the partial differential equation is dynamically unstable.

Example 8.6. *Dispersive waves.*

Consider

$$u_t = \frac{\partial^{2m+1} u}{\partial x^{2m+1}} + \frac{\partial^{2m} u}{\partial x^{2m}} + l.o.t.,$$

where m is a non-negative integer. For the simplest case $u_t = u_{xxx}$, we have

$$\widehat{u_t} = \widehat{\beta u_{xxx}}, \quad \text{or} \quad \widehat{u_t} = \beta(i\omega)^3 \hat{u} = -i\omega^3 \hat{u} \,,$$

and the solution of this ODE is

$$\hat{u}(\omega,t) = \hat{u}(\omega,0)\, e^{-i\omega^3 t} \,.$$

Therefore

$$\|u\|_2 = \|\hat{u}\|_2 = \|\hat{u}(\omega,0)\|_2 = \|u(\omega,0)\|_2 \,,$$

and the solution to the original PDE can be expressed as

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x}\, \hat{u}_0(\omega)\, e^{-i\omega^3 t}\, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega(x-\omega^2 t)}\, \hat{u}_0(\omega)\, d\omega \,. \end{aligned}$$

Evidently, the Fourier component with wave number ω propagates with velocity ω^2, so waves mutually interact but there is no diffusion.

Example 8.7. *PDEs with higher order derivatives.*

Consider

$$u_t = \alpha \frac{\partial^{2m} u}{\partial x^{2m}} + \frac{\partial^{2m-1} u}{\partial x^{2m-1}} + l.o.t.,$$

where m is a non-negative integer. The Fourier transform yields

$$\widehat{u_t} = \alpha(i\omega)^{2m} \hat{u} + \cdots = \begin{cases} -\alpha\omega^{2m}\hat{u} + \cdots & \text{if } m = 2k+1, \\ \alpha\omega^{2m}\hat{u} + \cdots & \text{if } m = 2k, \end{cases}$$

hence

$$\widehat{u} = \begin{cases} \hat{u}(\omega,0)\, e^{-\alpha i \omega^{2m} t} + \cdots & \text{if } m = 2k+1, \\ \hat{u}(\omega,0)\, e^{\alpha i \omega^{2m} t} + \cdots & \text{if } m = 2k\ , \end{cases}$$

such that $u_t = u_{xx}$ and $u_t = -u_{xxxx}$ are dynamically stable, whereas $u_t = -u_{xx}$ and $u_t = u_{xxxx}$ are dynamically unstable.

Chapter 9

The Laplace transform

The Fourier transform is for the entire space $(-\infty, \infty)$ while the Laplace transform is for half space $(0, \infty)$ such as time variable $t > 0$. The Laplace transform is defined as

$$L(f)(s) = F(s) = int_0^\infty f(t)e^{-st}\, dt, \tag{9.1}$$

where s is in a complex number. A necessary condition for existence of the integral is that f must be locally integrable on $[0, \infty)$.

Example 9.1. *Find the Laplace transform of* $f(t) = 1$, $f(t) = t^2$, $f(t) = t^\alpha$, *and* $f(t) = \alpha t$,

We apply the facular to get

Appendix A

ODE Review

A.1 First order ODEs

We review the solution or ODEs. We start with the first order linear and homogeneous PDE

$$\frac{dy}{dx} + p(x)y(x) = 0. \tag{A.1}$$

For a more general ODE $a(x)\frac{dy}{dx} + b(x)y(x) = 0$ we can divide by $a(x)$ assuming it is not zero to get $\frac{dy}{dx} + \frac{b(x)}{a(x)}y(x) = 0$. If $a(x) = 0$ at some places, the differential equation is singular since there is no derivative involved.

If we rewrite the ODEs as

$$\frac{dy}{y} = -p(x)dx, \tag{A.2}$$

and integrate on both sides, we get

$$\int \frac{dy}{y} = -\int p(x)dx + C, \qquad \log|y(x)| = C - \int p(x)dx.$$

The solution can be written as

$$y(x) = Ce^{-\int p(x)dx}. \tag{A.3}$$

For a non-homogeneous ODE

$$\frac{dy}{dx} + p(x)y(x) = g(x), \tag{A.4}$$

we can multiply a function $\mu(x)$, call an integrating factor so that the ODE can become something $\frac{d}{dx}(sth.) = f(x)$ and can be integrated easily. In other words, we wish to have

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y(x) = g(x)\mu(x). \tag{A.5}$$

We wish the left hand side becomes $\frac{d}{dx}(\mu(x)y(x)) = g(x)\mu(x)$, then the solution would be

$$y(x) = \frac{1}{\mu(x)}\left(\int g(x)\mu(x) + C_1\right).$$

The left hand side in (A.5) is the same as the left hand side of the ODE. This leads to

$$\mu y' + \mu' y = \mu y' + \mu p y, \qquad \text{or} \qquad \mu' = \mu p.$$

We get $\mu(x) = C_2 e^{\int p(x)dx}$. Plug this into (A.6), we get the solution

$$\begin{aligned} y(x) &= \frac{1}{C_2}e^{-\int p(x)dx}\left(\int g(x)C_2 e^{\int p(x)dx} + C_1\right) \\ &= e^{-\int p(x)dx}\left(\frac{C_1}{C_2} + \int g(x)e^{\int p(x)dx}\right) \\ &= e^{-\int p(x)dx}\left(C + \int g(x)e^{\int p(x)dx}\right) \end{aligned} \tag{A.6}$$

Example A.1. *Solve* $y'(x) - y(x) = 2$.

In this example, $p(x) = -1$, $g(x) = 2$, the solution is

$$y(x) = e^{\int 1dx}\left(C + \int 2e^{\int(-1)dx}\right) = e^x\left(C - 2e^{-x}\right).$$

Note that, for this problem we can also use the undetermined coefficients method to find the particular solution.

A.2 Second order linear and homogeneous ODE with constant coefficients

The ODE has the form

$$ay''(x) + b(x)y' + cy = 0. \tag{A.7}$$

The corresponding characteristic polynomial is defined as

$$a\lambda^2 + b\lambda + c = 0 \tag{A.8}$$

whose discriminant and roots are

$$b^2 - 4ac \begin{cases} > 0 & \text{there are two distinct roots } \lambda_1, \lambda_2, \\ = 0 & \text{there is one double root, } \lambda_1 = \lambda_2 = \lambda, \\ < 0 & \text{there are two complex roots } \lambda_1 = \alpha + \beta i,\ \lambda_2 = \alpha - \beta i,\ i = \sqrt{-1}. \end{cases}$$

The solution to the ODE are

- $b^2 - 4ac > 0$, $y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
- $b^2 - 4ac = 0$, $y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$.
- $b^2 - 4ac < 0$, $y(x) = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right)$.

A.3 Useful trigonometric formulas

There formulas are very useful in Fourier analysis

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right) \tag{A.9}$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right) \tag{A.10}$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \sin(\alpha-\beta)\right) \tag{A.11}$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\left(\cos(\alpha+\beta) - \cos(\alpha-\beta)\right) \tag{A.12}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} \tag{A.13}$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}. \tag{A.14}$$

A.4 ODE solutions to the Euler's equations

A second order Euler's equation has the following form

$$x^2 y'' + \alpha x y' + \beta y = 0. \tag{A.15}$$

An nth order Euler's equation has the following form

$$x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_1 x y + a_0 y = 0. \tag{A.16}$$

For a first order Euler equation $xy' + \alpha y = 0$, we have $\frac{y'}{y} = -\frac{\alpha}{x}$ and the solution is $\log|y(x)| = -\alpha\log|x| + C$ or $y = C|x|^{\alpha}$.

For a second order Euler's equation, we look for the solution of the form of $y(x) = x^r$, $y'(x) = rx^{r-1}$ and $y'(x) = r(r-1)x^{r-2}$. The ODE then becomes

$$x^2 r(r-1)x^{r-2} + \alpha x r x^{r-1} + \beta x^r = 0, \tag{A.17}$$

which leads to an indicial equation

$$r(r-1) + \alpha r + \beta = 0. \tag{A.18}$$

There are three cases corresponding to different general solutions.

1. Two distinct roots, r_1 and r_2. The solution then is

$$y(x) = C_1|x|^{r_1} + C_2|x|^{r_2}.$$

2. One repeated root r_1. The solution is

$$y(x) = C_1|x|^{r_1} + C_2 (\log|x|)|x|^{r_1}.$$

3. A complex pair $r = a \pm ib$, the solution is

$$y(x) = |x|^a \left(C_1 \cos(b \log|x|) + C_2 \sin(b \log|x|)\right).$$

Example A.2. *Solve the ordinary differential equation* $x^2y'' + 2xy' - 6y = 0$.

Solution: The indicial equation is $r(r-1) + 2r - 6 = 0$, or $r^2 + r - 6 = (r+3)(r-2) = 0$. Its roots are $r_1 = -3$ and $r_2 = 2$. The general solution is

$$y(x) = C_1|x|^{-3} + C_2|x|^2 = \frac{C_1}{|x|^3} + C_2x^2.$$

Example A.3. *Solve the ordinary differential equation* $x^2y'' + 3xy' + 10y = 0$.

Solution: The indicial equation is $r(r-1) + 3r + 10 = 0$, or $r^2 + 2r + 10 = 0$. The solutions are $r = -1 \pm 3i$. The general solution is

$$y(x) = |x|^{-1} \left(C_1 \cos(3 \log|x|) + C_2 \sin(3 \log|x|)\right).$$

Example A.4. *Solve the ordinary differential equation* $x^2y'' + 3xy' + y = 0$.

The indicial equation is $r(r-1) + 3r + 1 = 0$, or $r^2 + 2r + 1 = 0$. There is one double root $r = -1$. The general solution is

$$y(x) = C_1|x|^{-1} + C_2 (\log|x|)|x|^{-1}.$$

Index