MA 780
Homework #4
Due April 8

1. This problem provides a way in computing some special functions or integrals. (a): Use the asymptotic (linear approximation, *i.e.*, $p = 1$) and matching approach to compute the singular integral $\int_0^\pi \frac{\sin x}{\sqrt{x}}\, dx$ accurately to the machine precision (that is, as accurate as possible). Note that the true value is 'fresnels(sqrt(2))*sqrt(2*pi) = 1.789662938968290' using Matlab, where *fresnels* is the Fresnel sine function.

   (b): Now use the sequence $r_n = \dfrac{\pi}{2^n}$ to calculate the integral: $I_n = \displaystyle\sum_{k=0}^{n-1} \int_{r_{k+1}}^{r_k} \frac{\sin x}{\sqrt{x}}\, dx$ to compute the integral until $\dfrac{|I_n - I_{n-1}|}{|I_n|} < tol$.

   (c): Compare the two methods to see which method that you think is better and why. **Hint:** You should call ('trapez()' in Matlab) or write a composite trapezoidal quadrature formula.

2. Consider the 'mid-point' and the 'trapezoidal' quadrature formulas on the square: $[0, h] \times [0, h]$. (a) Find the algebraic precision using polynomials in two dimensions including constant, linear, *bi-linear*, quadratic and so on. (b): Find error estimates of the quadrature formulas for approximating $\iint_R f(x, y) dx dy$ in terms of h. **Hint:** Use the Taylor expansion in 2D.

   (c): Repeat the process for the right triangle T_h determined by the three vertices: $(0, 0)$, $(h, 0)$, and $(0, h)$.

3. Use the Monte Carlo method to evaluate the integral $\iint_\Omega \sin(kx) \cos(ky) dx dy$, where $\Omega = [-1,\ 1] \times [-1,\ 1]$ for $k = 1$, $k = 10$, and $k = 100$. (a): Find the true value of the integration. (b): Implement the Monte Carlo method with 500 sample points for 10 runs and then average the results and compare with the true values. (c): Analyze your computed results and provide some suggestions or conclusions.

   Hint: You need to either transform the domain or the randomly generated numbers.

4. Convert the IVP problem

$$y''' + \beta(1 - y^2)(y')^2 + \cos y = 0, \quad y(0) = 1, y'(0) = -1, y''(0) = 1$$

   to a first order system.

5. Consider one of two linear multi-step methods (choose one)

$$\begin{aligned} y_{n+1} &= \alpha y_n + \frac{h}{2}\Big(2(1-\beta) f_{n+1} + 3\beta f_n - \beta f_{n-1} \Big) \\ \text{or} \quad y_{n+1} &= \alpha y_{n-1} + \frac{h}{2}\Big(2(1-\beta) f_{n+1} + 3\beta f_n - 2\beta f_{n-1} \Big). \end{aligned}$$

   (a) Analyze consistency and order of the methods; determining the values α^* and β^* for which the resulting method has maximal order.

   (b) Study the zero-stability of the method with $\alpha = \alpha^*$ and $\beta = \beta^*$; write its characteristic polynomial and using MATLAB, draw its region of absolute stability in the complex plane.

6. Consider the two-stage of RK method (it is also called the improved Euler method):

$$\begin{aligned} K_1 &= f(x_n, y_n) \\ K_2 &= f(x_{n+1}, y_n + hK_1) \\ y_{n+1} &= y_n + \frac{h}{2}(K_1 + K_2) \end{aligned}$$

(a) Show the local truncation error is order of h^2.

(b) Discuss the zero and absolute stability of the algorithm. Find or plot the stability region.

7. Referring to mymovie.m and chemistry.m on the class web-page, use Matlab ODE-suite to solve the following IVP

$$\begin{aligned} y_1' &= a - (b+1)y_1 + y_1^2 y_2 \\ y_2' &= by_1 - y_1^2 y_2. \end{aligned}$$

Try the parameters $(a, b) = (1, 3), (1, 10), (1, 20), (5, 1)$, $T = 50, 500$. Plot the history of the solution and phase plot. Find all the equilibriums and classify them (stable, unstable, center etc.). Do you think this problem is stiff? A stiff problem is that the eigenvalues of the linearized system have very different magnitude.

8. **Extra Credit:** This problem illustrates how we can combined different methods to solve problems. Implement the Crank-Nicolson scheme for the Lorenz equations using a non-linear system of equations solver or your own code.

$$\begin{aligned} x' &= -\tau x + \tau y \\ y' &= -xz + rx - y \\ z' &= xy - \beta z. \end{aligned}$$

Try you method at least with one set of data $r = 100, \tau = 10, \beta = 8/3$, final time $T = 20$ and an initial data $[1,\ 0,\ 0]$. Plot the history of the solution $x(t), y(t), z(t)$, and phase plot.

Apply one step Richardson extrapolation to the final solution and compare with that of direct computation and provide a posterior error estimate.

Try several typical sets of data. Write a report on your algorithm and analysis, compare your numerical results with those from literature; and try to interpret your results.