MA 780
Homework #3
Due March 4

1. (a): Consider a triangle whose vertices are $\mathbf{x}_i = (x_i, y_i)$, $i = 1, 2, 3$. Assume that the area of the triangle is no-zero. Show that there is a unique linear function $u_h(x, y) = a_{0,0} + a_{1,0}x + a_{0,1}y$ that interpolates a function $f(x, y)$ at three vertices. Thus, the interpolation can be written as $u_h(\mathbf{x}) = \sum_{i=1}^{3} f(\mathbf{x}_i)\phi_i(\mathbf{x})$, where $\phi_i(\mathbf{x})$ is the basis functions satisfying $\phi_i(\mathbf{x}_j) = \delta_i^j$.

   (b): **Optional:** Assuming that $f(\mathbf{x}) \in C^2$ and h is the longest side of the triangle, show that $|f(\mathbf{x}) - u_h(\mathbf{x})| \le Ch^2$. So the interpolation is second order accurate.

2. Fin and list the Newton-Cotes coefficients, closed and open, for $0 \le n \le 6$. Which ones are stable? **Hint:** Use a symbolic software package, for example, Maple.

3. (a) Show that the Simpson quadrature rule has algebraic precision 3 by testing the polynomial basis function $\{1, x, x^2, x^3, x^4\}$. Therefore the Simpson quadrature rule is exact for any polynomial of $p_k(x)$ of degree $k \le 3$.

   (b) Consider the polynomial interpolation

$$p_3(a) = f(a), \qquad p_3(b) = f(b),$$
$$p_3(c) = f(c), \qquad p_3'(c) = f'(c), \qquad c = \frac{a+b}{2}.$$

   We know that

$$f(x) - p_3(x) = \frac{f^{(4)}(\xi(x))}{4!}(x-a)(x-c)^2(x-b)$$

   assuming $f(x) \in C^4(a, b)$. Use the error estimate above to show that

$$\int_a^b f(x)dx - \frac{b-a}{6}\left(f(a) + 4f(c) + f(b)\right) = -\frac{(b-a)^5}{2880} f^{(4)}(\eta).$$

   (c) Using the error estimate above to show the following error estimate for the composite Simpson rule:

$$\int_a^b f(x)dx - S_n = -\frac{b-a}{2880} f^{(4)}(\eta)\, h^4.$$

4. Assume that $\frac{f^k(x)}{k!}$ are all $O(1)$ quantities for all k's. If we wish to approximate $\int_0^1 f(x)dx$ with different quadrature methods such that the error is less than 10^{-10}, estimate the smallest n that is needed for the following methods:

   (a) Composite trapezoidal formula.

   (b) Composite Simpson formula.

   (c) Romberg method.

5. (a) Find the coefficients of the following quadrature

$$\int_0^1 f(x)dx \approx \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0).$$

(b) What is the algebraic precision of the quadrature formula?

(c) Can you give an error estimate?

6. Find x_1 and x_2 such that the following quadrature

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{3}\left(f(-1) + 2f(x_1) + 3f(x_2)\right)$$

has as high algebraic precision as possible.

7. Reformulate the following integrals to normal integrals.

(a) $\displaystyle\int_0^1 x^2 \log x dx$.

(b) $\displaystyle\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$.

(c) $\displaystyle\int_0^{\pi} \frac{\sin x}{x^{\mu}} dx, \quad 0 < \mu < 2.$

8. Implement Romberg integration to approximate $\int_a^b f(x)dx$:

(a) $a = 0,\ b = \dfrac{\pi}{2}.\ f(x) = \dfrac{5e^{2x}}{e^{\pi} - 2}$. Note that $\int_0^{\pi/2} f(x)dx = \frac{5(e^{\pi}-1)}{2(e^{\pi}-2)}$.

(b) $a = 0,\ b = 1,\ f(x) = x^2 \log(x)$, where $\log(x)$ is the natural logarithm. You should be able to find the exact solution by integration by parts. (**Hint:** Use $\log(x) \approx \log(x + \epsilon)$ to avoid the singularity. Take $\epsilon = 10^{-16}$, for example).

(c) Find the circumference (its length) of the ellipse $x^2 + \frac{y^2}{4} = 1$. (**Hint:** $L = \oint ds$. Write down the ellipse in terms of the parametric form $x = \cos t$; $y = 2\sin t$, $ds = \sqrt{dx^2 + dy^2}$).

Use the computed approximations to estimate the errors. Analyze the results and compare with the exact error if possible.