MA 780: Test 1

Name: **Student ID:**

<u>**Show all your work, partial credit(s) will be given for partial results**.</u>

1. Short questions and fast evaluations. No proof and derivations are needed.

   (a) Given (x_0, y_0), (x_1, y_1), $\cdots$, (x_n, y_n), $x_i \neq x_j$ if $i \neq j$, and let $l_i(x)$ be the Lagrange base polynomial. Find the values of

   $$\sum_{i=0}^{n} l_i(x) = 1 \qquad \sum_{i=0}^{n} l_i(x) x_i^k = x^k \qquad \sum_{i=0}^{n} l_i(x) p_k(x_i) = p_k(x)$$

   where $k \leq n$, $p_k(x)$ is a polynomial of degree of $k \leq n$. For any polynomial of degree of $k \leq n$, the interpolation is exact since $|E| = \frac{f^{n+1}(\xi)}{(n+1)!} = 0|!$ That is, $\sum_{i=0}^{n} p_k(x_i) l_i(x) = p_k(x)$. We can take $p_k(x) = 1$, or $p_k(x) = x^k$, or any polynomial of degree of $k \leq n$.

   (b) $f[0, 1, 2, \cdots, 7] = 7!/7! = 1$ if $f(x) = x^7 + 128x^3 + 1$ since $f[x_0, x_1, \cdots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!} = |\frac{f^{(7)}(\xi)}{(6+1)!}$.

   (c) $f[0, 1, 2, \cdots, 7, 8] = 0$ if $f(x) = x^7 + 3x^3 + 76$ for the same reasons.

   (d) * $f[x_0, x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n-1}, x_n] = \frac{1}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)}$ from the formula $f[x_0, x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{n-1}, x_n] = \sum \frac{f(x_i)}{\omega'_{n+1}(x_i)}$ when $f(x) = l_i(x)$ since $l_i(x_j) = \delta_i^j$ and $\omega'_{n+1}(x_i) = \prod_{j=0, j\neq i}^{n} (x_i - x_j)$ from the formula $f[x_0, x_1, \cdots, x_n] = \sum_0^n \frac{f(x_i)}{\omega'_{n+1}(x_i))}$.

2. (i) Can you find a linear function $p_1(x)$ such that $p_1'(0) = 0$ and $p'(1) = 0$? Does the solution exist? If so, is it unique? (ii) Use the divided difference to find the polynomial to interpolate $f(0) = 1$, $f(1) = 2$, $f'(0) = 0$, $f''(0) = 1$. (iii) Find the error estimate (as accurate as possible) for the interpolation assuming $f(x) \in C^3(0, 1)$.

   Solution: We can see that $p_1(x) = C$ satisfies the condition for any constant C. So it exists but not unique. The divided difference table is, note that $f[0, 0, 0] = f''(0)/2$,

0	1			
0	1	0		
0	1	0	1/2	
1	2	1	1	1/2

Thus, the solution is $p_3(x) = 1+x^2/2+x^3/2$. An error estimate is $|E| = |\frac{f^{(4)}(\xi)}{4!}x^3(x-1)| \le M_{xxxx} \le \frac{27}{24\times 4^4}$. The maximum is reached at $x = 3/4$.

3. Use the Lagrange polynomial interpolation to find a linear transform $\xi = \alpha x + \beta$ that transforms the interval $[a,\ b]$ to $[-1, 1]$, that is $\xi(a) = -1$ and $\xi(b) = 1$. Find the basis functions of Hermite interpolation $\phi_i(x)$ such that $\phi_i(x_i) = \delta_i^j$ and $\phi_i'(x_j) = 0$, and $\psi_i(x)$ such that $\psi_i(x_j) = 0$ and $\psi_i'(x_j) = \delta_i^j$, where $i, j = 1, 2$, $x_1 = -1$ and $x_2 = 1$.

   Solution: Use the Lagrange interpolation, we have $\xi(x) = \dfrac{x-b}{a-b}(-1) + \dfrac{x-a}{b-a}1 =$. To find the Hermite interpolation $\phi(x)$ at $x = -1$, we know that $\phi(x) = (x-1)^2(a(x+1)+b$ since $x = 1$ is a double root. Set $x = -1$, we get $b = 1/4$. $\phi(x) = \frac{(x-1)^2}{4}(\bar{a}(x+1)+1)$. Take the derivative, we get $\phi'(x) = \frac{(x-1)}{2}(\bar{a}+1) + \frac{(x-1)^2}{4}\bar{a}$. Plug in $x = -1$ we get $\bar(a) = 1$ and the function is $\frac{(x-1)^2(x+2)}{4}$. Similarly, for the second type basis function, we have $\psi(x) = (x-1)^2(a(x+1)+b$; $\psi(-1) = 0$, we get $b = 0$. By taking the first order derivative and plugging $x = -1$, we can easily get $\psi(x) = (x-1)^2(x+1)/4$.

4. Let $S_3(x) \in C^2[x_0, x_n]$ be the cubic spline given the nodal points $x_0, x_1, \cdots, x_n$. (i) Derive the degree of the freedom (DOF) of $S_3(x)$. (ii) If $S_3(x)$ interpolates a function $f(x)$, that is, $S_3(x_i) = f(x_i)$, $i = 0, 1, \cdots, n$, how many DOF is left? (iii) Express $S_3(x)$ in terms of its second moments $S''(x_i) = M_i$. (iv) Write down the system of equations for M_i with equally spaced nodes and a periodic boundary condition.

   Solution: For a $S_3(x)$ on the given mesh, the DOF $= 4n$. The continuity of the function, the first and second order derivative at interior nodes have $3(n-1)$ constraints. Thus the remaining DOF is $4n - 3(n-1) = n+3$. If we put the interpolation condition $S_3(x_i) = f(x_i)$, then DOF=2.

   Using the Lagrange interpolation for the second moments, we have

$$S_3''(x) = -\frac{x-x_j}{h}M_{j-1} + \frac{x-x_{j+1}}{h}M_j, \qquad x_{j-1} \le x \le x_j$$

   For period boundary condition, from homework we know that

$$\begin{bmatrix} 2 & \frac{1}{2} & & & & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} & & & \\ & \frac{1}{2} & 2 & \frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & & & & \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ U_3 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}.$$

5. Given an algorithm $A(h)$ that satisfies $A(h) = A_0 + Ch^3 + O(h^5)$. Find the formula of the one-step Richardson extrapolation using $h/2$. Derive a posterior error estimate, that is, find the error of the algorithm for approximating A_0 without knowing the true value A_0.

   Solution: We have $A(h/2) = A_0 + Ch^3/8 + O(h^5)$. We want to get A_0 with $O(h^5)$ error order, so we do

$$A(h/2) - A(h)/8 = A_0 - A_0/8 + O(h^5) \Longrightarrow A_0 \approx \frac{A(h/2) - A(h)/8}{1 - 1/8}.$$

   Next Subtract $A(h/2)$ from above to get a posterior error estimate

$$A_0 - A(h/2) \approx \frac{A(h/2) - A(h)/8}{1 - 1/8} - A(h/2) = \frac{A(h/2) - A(h)}{7}.$$

   where A_0 is the unknown true value.

6. **Extra credit.** Given a set of data: $(x_i, f(x_i))$, $i = 0, 1, \cdots, n$, and $f'(x_0)$, $x_i \neq x_j$ if $i \neq j$. Show that there is a unique piecewise quadratic function $u_h(x) \in C^1$ that interpolates the data. Also assume that $f(x) \in C^3$, find an error estimate of the interpolation assuming equally spaced nodes, $x_{i+1} - x_i = h$, $i = 0, 1, \cdots, n-1$.

   Solution: In the first interval $[x_0, x_1]$. the interpolation is unique and the error satisfies $|E_1| \leq |\frac{f'''(\xi)}{6}(x - x_0)^2(x - x_1)|$. From the interpolation, we can get an approximation of $f'(x_1)$ and an error estimate. From the information, we can get the interpolation in $[x_1, x_2]$ and an error estimate. We can repeat the process until the last interval.