Notes on piecewise quadratic interpolation.

Problem: Given (x_i, y_i), $i=0, 1, \ldots n$, $n>0$, can we find a piecewise quadratic polynomial in $C[x_0, x_n]$ such that $u_h^2(x_i) = y_i$, $i=0, 1, \ldots n$.

Answer: Yes, but it is not unique!

Construct a piecewise quadratic polynomial.

Define $\varphi_i(x)$ as a quadratic piecewise base polynomial such that

$$\varphi_i(x_j) = \delta_i^j, \quad \varphi_i\left(\frac{x_j + x_{j+1}}{2}\right) = 0, \quad \begin{array}{l} i=0, 1, \ldots, n \\ j=0, 1, \ldots, n-1. \end{array}$$

$$\varphi_i(x) = \begin{cases} 0 & x_0 \le x \le x_{i-1} \\ A\left(x - \frac{x_{i-1} + x_i}{2}\right)(x - x_{i-1}) & x_{i-1} \le x \le x_i \\ B\left(x - \frac{x_i + x_{i+1}}{2}\right)(x - x_{i+1}) & x_i \le x < x_{i+1} \\ 0 & x_{i+1} \le x \le x_n \end{cases}$$

Define $\overline{\varphi}_i(x)$ such that $\overline{\varphi}_i(x_j) = 0$, $j=0, 1, \ldots, n$

$$\overline{\varphi}_i\left(\frac{x_{i-1} + x_i}{2}\right) = 1, \quad \overline{\varphi}_i\left(\frac{x_j + x_{j+1}}{2}\right) = 0, \quad j=0, \ldots, n-1.$$

$$\varphi_i(x) = \begin{cases} 0 & x_0 \le x \le x_{i-1} \\ C(x - x_i)(x - x_{i+1}) & x_i \le x \le x_{i+1} \\ 0 & x_{i+1} \le x \le x_n \end{cases}$$

Then

$$u_h^2(x) = \sum_{i=0}^{n} y_i \varphi_i(x) + \sum_{i=1}^{n} \beta_i \bar{\varphi}_i(x)$$

is a piecewise quadratic interpolation for any choice $\{\beta_i\}_{i=1}^{n}$, we have

$$u_h^I(x_k) = \sum_{i=0}^{n} y_i \varphi_i(x_k) + \sum_{i=1}^{n} \beta_i \bar{\varphi}_i(x_k)$$

$$= y_k$$

We need other n conditions to uniquely determine the piecewise quadratic polynomial.

It is also true for piecewise cubic interpolations in $C[x_0, x_1]$, we need about $2n+?$ additional conditions.

We can set $\beta_i = 0$ to get

$$u_h^I(x) = \sum_{i=0}^{n} y_i \varphi_i(x)$$

or more reasonably $\beta_i = \frac{y_i + y_{i-1}}{2}$, $i = 1, 2, \ldots n$

to get $$u_h^I(x) = \sum_{i=0}^{n} y_i \varphi_i(x) + \sum_{i=1}^{n} \frac{y_i + y_{i-1}}{2} \bar{\varphi}_i(x).$$