Q.1 What is a Romberg integration?

Apply the Richardson Table to the composite trapezoidal rule with an initial h. Simpson's rule is included.

2. What is a posterior error analysis?

$$\left| I_n(f) - \int_a^b f(x)dx \right| \le \frac{|f''(\eta)|}{12} h^2$$

prior

A prior analysis

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Posterior error analysis

We use computed solutions to find 'the error' without knowing true solution; and use the results to improve/adjust the computation.
- adaptive time step
- adaptive mesh refinement (AMR)
- ~~better~~ derivative computation → FEM

For the Romberg integration. Posterior error analysis.

Given a tolerance $Tol = 10^{-8}$.

h	$O(h^2)$	$O(h^4)$	$O(h^6)$
	A_{00}		
	A_{10}	A_{11}	
	A_{20}	A_{21}	A_{22}

We know

$$A_{i,k} \approx I(f) + C h^{2k+2}$$

$$A_{i+1,k} \approx I(f) + C \frac{h^{2k+2}}{2^{2k+2}}$$

$$\frac{A_{i,k} - I(f)}{A_{i+1,k} - I(f)} \approx 4^{k+1}$$

Solve for $I(f)$, we get

$$I(f) = \frac{4^{k+1} A_{i+1,k} - A_{i,k}}{4^{k+1} - 1}$$ One step Richardson

$$I(f) - A_{i+1,k} = \frac{4^{k+1} A_{i+1,k} - A_{i,k}}{4^{k+1} - 1} - A_{i+1,k}$$

$$= \frac{A_{i+1,k} - A_{i,k}}{4^{k+1} - 1}$$ $\le Tol$ then stop otherwise

$$R(f) \approx \frac{|I(f) - A_{i+1,k}|}{|A_{i+1,k}|} = \left| \frac{A_{i+1,k} - A_{i,k}}{(4^{k+1} - 1)|A_{i+1,k}|} \right| \le Tol$$

x
x x
x x x
x x x

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Evaluate singular/unbounded Integration
- $f(x)$ is discontinuous. $f(x) \in L^2(a,b)$
- $f(x)$ is singular (unbounded)

e.g. $f(x) = x \log x$ in $(0,1)$ $\int_0^1 x \log x \, dx$

$f(x) = \frac{g(x)}{(x-\alpha)^\mu}$, $0<\mu<1$, $g(x) \in C(0,1)$

Complex analysis, integral equation

fractional derivatives.

$0<\alpha<1$, $\frac{\partial^\alpha f(x)}{\partial x^\alpha} \triangleq \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f'(t)\,dt}{(x-t)^\alpha}$

- $\int_a^b f(x)dx$, $b=\infty$ or $a=-\infty$

ABC: Artificial boundary condition

PML: perfect matched layer

Wave scattering

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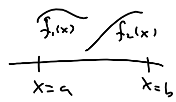
I: $f(x)$ is discontinuous but bounded

$$f(x) = \begin{cases} f_1(x) & a \le x < \alpha \\ f_2(x) & \alpha < x \le b \end{cases}$$

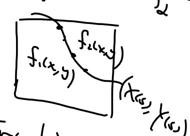


Soln. $\int_a^b f(x)dx = \int_a^\alpha f_1(x)dx + \int_\alpha^b f_2(x)dx$

Can we apply the composite trapezoidal Rule

less accurate!

in 2D:



Front Tracking, level set method.

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II Removable Singularity. (analytic, and then numerical) Change variables

Integration by parts ⟹ Remove singularities

e.g. $\int_0^1 \frac{f(x)}{\sqrt[n]{x}} dx$, $n \ge 2$, $f(x)$ is integrable

set $t = \sqrt[n]{x} \Rightarrow t^n = x$ $\quad n t^{n-1} dt = dx$

$$\int_0^1 \frac{f(x)}{\sqrt[n]{x}} dx = \int_0^1 \frac{f(t^n)}{t} n t^{n-1} dt$$

$$= \int_0^1 f(t^n) n t^{n-2} dt$$ regular integration

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Ex 2 $\int_a^b (\log x)^m P_k(x) dx$ $\quad P_k(0)=0$

Integration by parts.

$$\int_0^b (\log x)^2 (a_2 x^2 + a_1 x)\, dx$$

$$= (\log x)^2 \left(\frac{a_2 x^3}{3} + \frac{a_1 x^2}{2}\right)\Big|_0^b - \int_0^b 2 \log x \frac{1}{x} \left(\frac{a_2 x^3}{3} + \frac{a_1 x^2}{2}\right) dx$$

$$= (\log b)^2 \left(\frac{a_2}{3} b^3 + \frac{a_1 b^2}{2}\right) - 2 \log x \left(\frac{a_2 x^3}{9} + \frac{a_1 x^2}{4}\right)\Big|_0^b + \int_0^b \left(\frac{2}{9} x^3 + \frac{a_1 x^2}{2}\right) dx$$

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Ex. $\int_0^1 \frac{\cos x}{\sqrt{x}}dx = 2\sqrt{x}\cdot\cos x\Big|_0^1 + \int_0^1 2\sqrt{x}\sin x\,dx$

Evaluate singular integrals by a limiting process.

Case 1: Can identify the degree of the singularity

$f(x) = \frac{\varphi(x)}{(x-a)^\mu}$, $0<\mu<1$, $\varphi(a)\neq 0$ $\quad \frac{1}{(x-a)^k}$ Pade

ref the application of a fractional derivative

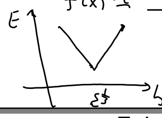
$\int_a^b f(x)dx = \left(\int_a^{a+\varepsilon} + \int_{a+\varepsilon}^b\right) f(x)dx = I_1(\varepsilon)+I_2(\varepsilon)$

Q1. How to choose ε?

Q2. How to evaluate $I_1(\varepsilon)$

$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ $\quad E \approx O(h^2) + O(\frac{\varepsilon}{h})$

$h = \varepsilon^{1/3}$ $\quad \varepsilon = 10^{-16}$ $\quad \varepsilon = 10^{-5}$



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$I_1(\varepsilon) = \int_a^{a+\varepsilon} \frac{\varphi(x)\,dx}{(x-a)^\mu} = \int_a^{a+\varepsilon} \sum_{k=0}^{p} \frac{\varphi^{(k)}(a)(x-a)^k}{k!\,(x-a)^\mu}dx$ + h.o.t.

$= \sum_{k=0}^{p} \frac{\varphi^{(k)}(a)\,(x-a)^{k-\mu+1}}{k!\,(k-\mu+1)}\Big|_a^{a+\varepsilon} + E(I_1(\varepsilon))$

computable, no singularity.

$|E(I_1(\varepsilon))| \leq \frac{|\varphi^{(p+1)}(\eta)|}{(p+1)!}\varepsilon^{p+2-\mu}$ $\quad 0<\varepsilon<1$

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$I_2(\varepsilon) = \int_{a+\varepsilon}^b \frac{\varphi(x)}{(x-a)^\mu}dx$, Composite Trapezoidal

$$E(I_2) = \begin{cases} \frac{|f''(\eta)|}{12}(b-a)h^2 \\ \frac{|f^{(4)}(\eta)|}{2880}(b-a)h^4 \end{cases}$$

The best ε is the one such that

$E_1(I_1) \approx E(I_2)$

$\max_{a\leq\eta\leq b}|f''(\eta)| = \max_{a\leq x\leq b}\left|\frac{\varphi''(x)}{(x-a)^{\mu+2}}\right| = \frac{1}{(x-a)^{\mu+2}} \sim \frac{1}{\varepsilon^{\mu+2}}$

$p=2$, $\quad \varepsilon^{4-\mu} \approx \frac{h^2}{12\,\varepsilon^{2+\mu}}$, $\quad h \approx \sqrt{12}\,\varepsilon^3$

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