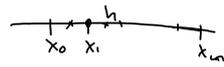


Composite quadrature rule
 effective implementation → recursive relation
 How do we choose n?
 Relative errors 0.200
 ↔ significant digits? 0.2
 10⁻⁷ often it means 7 significant digits
 - Acceleration and automatic integration machine learning
 Richardson extrapolation → Applied to quadrature
 Romberg technique
 - Error analysis Discrete mean value theorem

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Composite trapezoidal 

$$\int_a^b f(x) dx = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} f(x) dx$$

$$= \frac{h}{2} (f(a) + f(b)) + h \sum_{i=1}^{m-1} f(x_i) = T_m$$
 Find the relation between T_m and T_{2m}

$$T_{2m} = \frac{h_m}{2} (f(a) + f(b)) + \sum_{i=1}^{m-1} f(x_{2i}) h_{2m} \quad h_{2m} = \frac{h_m}{2}$$

$$= \frac{h}{4} (f(a) + f(b)) + \frac{h}{2} \sum_{i=1}^{m-1} f(x_{2i}) + \sum_{i=0}^{m-1} f(x_{2i+1}) h_{2m}$$

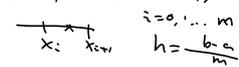
$$= \frac{1}{2} \left(\frac{h}{2} (f(a) + f(b)) + h \sum_{i=1}^{m-1} f(x_{2i}) \right) + \frac{1}{2} H_m$$

$$= \frac{1}{2} (T_m + H_m)$$

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function I = trapz C(a, b, k, f)
 h = b - a;
 I = (feval(f, a) + feval(f, b)) * h / 2;
 for l = 1:k
 h1 = h/2; HM = 0; M = 2^(l-1);
 for j = 1:M
 HM = HM + feval(f, a + (2*j-1)*h1);
 end
 HM = HM * h1; I = (I + HM) / 2;
 end
 Ex: f(x) = e^x, $\int_a^b f(x) dx = e^b - e^a$

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Composite Simpson Formula. $x_i = a + ih$,
 staggered grid 
 $x_{i+2} = x_i + \frac{h}{2}$, $i=0, \dots, m$
 Relation between composite trapezoidal and Simpson formulas.

$$I = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} f(x) dx = \sum_{i=0}^{m-1} \frac{h}{6} [f(x_i) + 4f(x_{i+1/2}) + f(x_{i+1})]$$

$$S_m = \frac{h}{6} \sum_{i=0}^{m-1} (f(x_i) + 4f(x_{i+1/2}) + f(x_{i+1}))$$

$$S_m = \frac{1}{3} T_m + \frac{2}{3} H_m$$
 Recall $T_{2m} = \frac{1}{2} T_m + \frac{1}{2} H_m \Rightarrow H_m = 2T_{2m} - T_m$

$$S_m = \frac{1}{3} T_m + \frac{2}{3} (2T_{2m} - T_m)$$

$$= \frac{4T_{2m} - T_m}{3}$$
 One step Richardson extrapolation

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Error estimates for composite quadratures
 $|T_m - I| \leq Ch^2$, $|S_m - I| \leq Ch^3$
 $\int_a^b f(x) dx = \int_a^{\eta} f(x) dx + \int_{\eta}^b f(x) dx$, $a \leq \eta \leq b$
 Discrete mean-value theorem
 If $u(x) \in C[a, b]$, $\{\delta_j\}_{j=0}^m$
 Then $\sum_{j=0}^m \delta_j u(x_j) = u(\eta) \sum_{j=0}^m \delta_j$, $\delta_j \geq 0$ for all j,
 $a \leq \eta \leq b$.
 Proof: Let $U_{max} = \max_{a \leq x \leq b} u(x)$, $U_{min} = \min_{a \leq x \leq b} u(x)$
 $U_{min} \sum_{j=0}^m \delta_j \leq \sum_{j=0}^m \delta_j u(x_j) \leq U_{max} \sum_{j=0}^m \delta_j$
 Condition has been used
 Next: Construct $U(x) = u(x) \sum_{j=0}^m \delta_j \in C[a, b]$
 $U_{min} \sum_{j=0}^m \delta_j \leq U(x) \leq U_{max} \sum_{j=0}^m \delta_j$
 From the mean value theorem, we know that there is η , $a \leq \eta \leq b$ such that
 $\sum_{j=0}^m \delta_j u(x_j) = u(\eta) \sum_{j=0}^m \delta_j$, $m \geq 1$

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Thm: For the composite formulas

Trapezoidal $E_n(f) = \int_a^b f(x) dx - T_n = -\frac{(b-a)^2 f''(\xi)}{12} h^2$

Simpson $E_n(f) = \int_a^b f(x) dx - S_n = -\frac{(b-a)^4 f^{(4)}(\xi)}{2880} h^4$

For the trapezoidal

$E_m(f) = \int_a^b f(x) dx - T_m = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} (f(x) - p_1(x)) dx$ linear interpolation

$= \sum_{i=0}^{m-1} -\frac{h^3}{12} f''(\eta_i)$ From Newton-Cotes 2/2 formula

$= -\frac{h^3}{12} \sum_{i=0}^{m-1} f''(\eta_i)$ $\eta_i = 1, i=0, \dots, m-1$

$= -\frac{h^3}{12} f''(\xi) \sum_{i=0}^{m-1} 1$ ≥ 0

$= -\frac{h^3}{12} f''(\xi) m = -\frac{h^2}{12} (b-a) f''(\xi)$ $h = \frac{b-a}{m}$

$\frac{E_m}{E_{2m}} \approx \frac{-\frac{h^2}{12} (b-a) f''(\xi)}{-\frac{(h/2)^2}{12} (b-a) f''(\xi)} \approx 4$ The quadrature is second order convergent or accurate.

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Richardson extrapolation technique

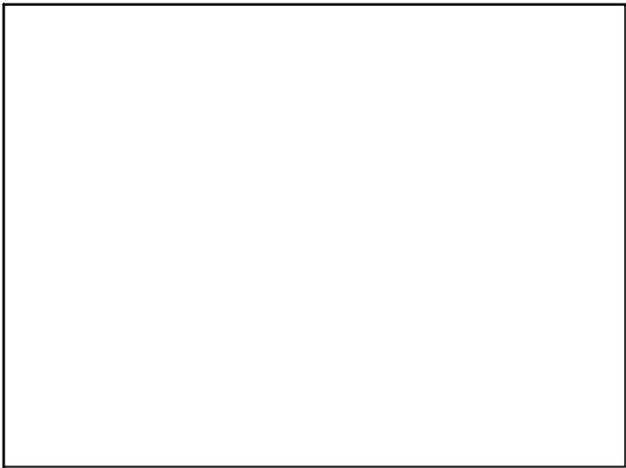
An acceleration technique

Applicable to many applications.

Motivation: get better accuracy using fewer points or function evaluation convergence

Ex: One step composite trapezoidal quadrature becomes composite Simpson's after one Richardson extrapolation $O(h^2) \downarrow O(h^4)$

Feb 19-4:09 PM



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