· Composite quadrature rule

Effective implementation → recursive relation

How do we choose n?

Relative errors $\leftrightarrow$ significant digits?

0.2000

0.2

10^{-7}, often it means 7 significant digits

— Acceleration and automatic integration

Richardson extrapolation → Applied to quadrature

machine learning

— Error analysis

Romberg technique,

Discrete mean value Theorem.

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Composite trapezoidal

$x_0 \quad x_1 \quad h \quad x_m$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$$

$$= \frac{h}{2}[f(a)+f(b)] + h\sum_{i=1}^{m-1} f(x_i) = T_m$$

Find the relation between T_m and T_{2m}

$$T_{2m} = \frac{h_{2m}}{2}[f(a)+f(b)] + \sum_{i=1}^{m-1} f(x_{2i})\,h_{2m} + \sum_{i=0}^{m-1} f(x_{2i+1})\,h_{2m} \qquad h_{2m} = \frac{h_m}{2}$$

$$= \frac{h}{4}[f(a)+f(b)] + \frac{h}{2}\sum_{i=1}^{m-1} f(x_{2i}) + \frac{h}{2}\sum_{i=0}^{m-1} f(x_{2i+1})$$

$$= \frac{1}{2}\left(\frac{h}{2}[f(a)+f(b)] + h\sum_{i=1}^{m-1} f(x_{2i})\right) + \frac{1}{2}H_m$$

$$= \frac{1}{2}(T_m + H_m)$$

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function I = trapzc(a, b, k, f)
    h = b-a;
    I = (feval(f, a) + feval(f, b))*h/2;
    for l = 1:k
        h1 = h/2;   HM=0;   M=2^(l-1);       2^(l-1)
        for j=1:M
            HM=HM+feval(f, a+(2*j-1)*h1);
        end
        HM=HM*h;   I = (I+HM)/2;
    end
```

Ex: $f(x) = e^x$, $\int_a^b f(x)\,dx = e^b - e^a$

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Composite Simpson Formula. $x_i = a + ih$, $i = 0, 1, \dots m$

staggered grid

$x_i \quad x_{i+1}$ $\quad h = \frac{b-a}{m}$

$x_{i+1/2} = x_i + \frac{h}{2}$,

MAC mark and cell

Relation between composite trapezoidal and Simpson formulas.

$$I = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} f(x)\,dx = \sum_{i=0}^{m-1} \frac{h}{6}\left[f(x_i) + 4f(x_{i+1/2}) + f(x_{i+1})\right]$$

$$S_m \triangleq \frac{h_m}{6}\sum_{i=0}^{m-1}[f(x_i) + f(x_{i+1})] + \frac{2}{3}\sum_{i=0}^{m-1} f(x_{i+1/2})\,h_m$$

$$S_m = \frac{1}{3}T_m + \frac{2}{3}H_m$$

Recall $T_{2m} = \frac{1}{2}T_m + \frac{1}{2}H_m \Rightarrow H_m = 2T_{2m} - T_m$

$$S_m = \frac{1}{3}T_m + \frac{2}{3}(2T_{2m} - T_m)$$

$$= \frac{4T_{2m} - T_m}{3}$$

One step Richardson extrapolation

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Error estimates for composite quadratures

$|T_m - I| \le Ch^2$, $\quad |S_m - I| \le \bar{C}h^3$

$$\int_a^b f(x)g(x)\,dx \overset{g(x)\ge 0}{\underset{g(x)\le 0}{=\!=}} f(\eta)\int_a^b g(x)\,dx \qquad a \le \eta \le b$$

Discrete mean-value theorem.

If $u(x) \in C[a,b]$, $\{\delta_j\}_{j=0}^m$ $\quad \delta_j \ge 0$ for all j's.

Then $\sum_{j=0}^m \delta_j u(x_j) = u(\eta)\sum_{j=0}^m \delta_j$. $\quad a \le \eta \le b$.

Proof: Let $U_{max} = \max_{a\le x\le b} u(x)$, $U_{min} = \min_{a\le x\le b} u(x)$

$$U_{min}\sum_{j=0}^m \delta_j \le \sum_{j=0}^m \delta_j u(x_j) \le U_{max}\sum_{j=0}^m \delta_j$$

Condition has been used

Next: Construct $U(x) = u(x)\sum_{j=0}^m \delta_j \in C[a,b]$

$$U_{min}\sum_{j=0}^m \delta_j \le U(x) \le U_{max}\sum_{j=0}^m \delta_j$$

From the mean value theorem, we know that there is η, $a \le \eta \le b$ such that

$$\sum_{j=0}^m \delta_j u(x_j) = u(\eta)\sum_{j=0}^m \delta_j. \qquad m \ge 1$$

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Thm: For the composite formulas 1

Trapezoidal $E_n(f) = \int_a^b f(x)\,dx - T_m = -\frac{(b-a)}{12} f''(\eta)\, h^2$

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Simpson $E_n(f) = \int_a^b f(x)\,dx - S_m = -\frac{(b-a)}{2880} f^{(4)}(\eta)\, h^4$

For the trapezoidal

linear interpolation

$$E_m(f) = \int_a^b f(x)\,dx - T_m = \sum_{i=0}^{m} \int_{x_i}^{x_{i+1}} (f(x)dx - p_1(x))\,dx$$

$$= \sum_{i=0}^{m-1} -\frac{h^3}{12} f''(\eta_i)$$ From Newton-Cotes error formula

$$= -\frac{h^3}{12} \sum_{i=0}^{m-1} f''(\eta_i)$$

$\delta_i = 1,\ i = 0, 1, \ldots m-1$

≥ 0

$$= -\frac{h^3}{12} f''(\eta) \sum_{i=0}^{m-1} 1$$

$$= -\frac{h^3}{12} f''(\eta)\, m = -\frac{h^2}{12}(b-a) f''(\eta)$$ $h = \frac{b-a}{m}$

$$\frac{E_m}{E_{2m}} \approx \frac{-\frac{h^2}{12}(b-a)\, f''(\eta_1)}{-\frac{(h/2)^2}{12}(b-a)\, f''(\eta_2)} \approx 4.$$

The quadrature is second order convergent or accurate.

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Richardson extrapolation technique

An acceleration technique

Applicable to many applications.

Motivation: get better accuracy convergence using fewer points or function evaluation

Ex: One step composite trapezoidal quadrature $O(h^2)$ becomes composite Simpson's after one Richardson extrapolation $O(h^4)$

$\downarrow$

$O(h^6)$

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