Q: Is $f(x)$ in C^2

$$f(x,t) = (x-t)_+^3 = \begin{cases} (x-t)^3 & \text{if } x>t. \\ 0 & \text{otherwise.} \end{cases}$$

B-splines. Yes!

piecewise cubic is $C^1[x_0, x_n]$, Hermite

2D interpolation.

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Localized splines. ① Local support.

② Non-negative. Stability

③ In C^2.

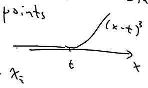
④ $\varphi_i(x_j) = \delta_i^j$ is not valid.

Construct B-splines.

A: from $f(x,t) = (x-t)_+^3 = \begin{cases} (x-t)^3 & \text{if } x>t \\ 0 & \text{otherwise} \end{cases}$

Expand the nodal points

$x_0, x_1, \ldots, x_n$



B-splines centered at x_i

x_{-3}, x_{-2}, x_{-1} $x_0, x_1, \ldots, x_n$ $x_{n+1}, x_{n+2}, x_{n+3}$

Then we can define B-splines at all x_i's by

$$N_i(x) = (x_{i+4} - x_i) \overbrace{f[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}]}^{f(x,t)}$$

with t-variable

If all x_i's are distinct, $i = -3, -2, \ldots 0, 1, \ldots n, n+1, n+2, n+3$

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$$f[x_0, x_1, \cdots, x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{\omega_{n+1}'(x_j)} \quad \text{Recall}$$

Thus,

$$N_i(x) = (x_{i+4} - x_i) \sum_{j=i}^{i+4} \frac{(x-x_j)_+^3}{\prod_{\substack{l=i \\ l\neq j}} (x_j - x_l)}$$

Thm: (i) $N_i(x) \in C^2[x_i, x_{i+4}]$

(ii) $N_i(x) \equiv 0$ if $x \le x_i$ or $x \ge x_{i+4}$

(iii) $N_i(x) \ge 0$.

(i) It's obvious if $x \le x_i$

If $x > x_{i+4}$, then $(x-x_j)_+^3 = (x-x_j)^3$

If $x_i \le x < x_{i+3}$, some of $(x-x_j)_+^3$ will be zero.

$$f[x_i, x_{i+1}, \cdots, x_{i+4}] = \left.\frac{\frac{\partial^4}{\partial t}(x-t)^3}{4!}\right|_{t=\xi} = 0!$$

$$f[\quad] = \frac{f^{(\ )}(\xi)}{(h\ \ )!}$$

(iii) $N_i(x) \ge 0$. is difficult!

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Ex: $x_i = i$, $i = 0, 1, \ldots$, $N_0(x)$

$$\frac{1}{4} N_0(x) = \begin{cases} 0 & x<0 \\ \dfrac{x^3}{(0-1)(0-2)(0-3)(0-4)} & 0 \le x \le 1 \\ \dfrac{x^3}{4!} + \dfrac{(x-1)^3}{(1-0)(1-2)(1-3)(1-4)} & 1 \le x \le 2 \\ \dfrac{x^3}{4!} + \dfrac{(x-1)^3}{(-3!)} + \dfrac{(x-2)^3}{(2-0)(2-1)(2-3)(2-4)} & \\ \dfrac{x^3}{4!} + \dfrac{(x-1)^3}{(-3!)} + \dfrac{(x-2)^3}{4} + \dfrac{(x-3)^3}{(3-0)(3-1)(3-2)(3-4)} & 2 \le x \le 3 \\ 0 & 4 \le x \end{cases}$$

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If we have repeated nodes, we use the derivative in the place of the divided differences.

$$f[x_0, x_0] = f'(x_0), \quad f[x_0, x_0, x_0] = \frac{f''(x_0)}{2!}$$

$$f[x_0, x_0, x_0, x_0] = \frac{f'''(x_0)}{6}.$$

For B-spline interpolation (x_i, y_i)

$$S_3(x) = \sum_{i=-3}^{n-1} \alpha_i N_i(x)$$

+ 2 more BC

$n-1-(-3)+1$
$= n-1+3+1$
$= n+1+\underline{\underline{2}}$

DOF = $n+3$

$$S_3(x_j) = \sum_{i=-3}^{n-1} \alpha_i N_i(x_j) \quad \neq \delta_i^j$$

It's sparse

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B-spline generated by a recursive relation.

piece wise constant $B_{i,1}(x) = \begin{cases} 1 & x_i \le x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases}$

piecewise linear. $B_{i,2}(x) = \dfrac{x - x_i}{x_{i+1} - x_i} B_{i,1}(x) + \dfrac{x_{i+2} - x}{x_{i+2} - x_{i+1}} B_{i+1,1}$

$$B_{i,k+1}(x) = \frac{x - x_i}{x_{i+k} - x_i} B_{i,k}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1,k}$$

$k = 1, 2, 3, 4. \longrightarrow$ B-splines.

We want to use B-splines if we wish to compute the values at a few points.

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Piecewise cubic interpolation in $C^1[x_0, x_n]$
Hermite

Problem: Given (x_i, y_i, y_i'). want to find (cubic spline)
a piecewise cubic function $u_h(x) \in C^1[x_0, x_n]$ such that

$u_h(x_i) = y_i$, $\quad u_h'(x_i) = y_i'$. $\quad C^2 \quad n+3$.

$$DOF = 4n - \underset{y_i}{(n+1)} - \underset{y_i'}{(n+1)} - \underset{C^0}{(n-1)} - \underset{C^1}{(n-1)} \quad C^1$$
$$= 0$$

It's wellposed.

Construct the local basis function

$\varphi_i(x)$ s.t. $\quad \varphi_i(x_j) = \delta_i^j$, $\quad \varphi_i'(x_j) = 0$ $\quad (n+1)$

$\bar{\varphi}_i(x)$ s.t. $\quad \bar{\varphi}_i'(x_j) = \delta_i^j$. $\quad \bar{\varphi}_i(x_j) = 0$

Hermite interpolation

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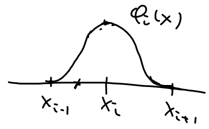
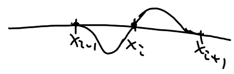


$$\varphi_i(x) = \begin{cases} 0 & x < x_{i-1} \\ (x-x_{i-1})^2 (a_i(x-x_i)+b_i) & \\ (x-x_{i+1})^2 (\bar{a}_i(x-x_{i+1})+\bar{b}_i) & \\ 0 & x > x_{i+1} \end{cases}$$



$$\bar{\varphi}_i(x) = \begin{cases} 0 & \\ (x-x_{i-1})(x-x_i)(\alpha_i(x-x_i)+\beta_i) & \\ (x-x_i)(x-x_{i+1})(\bar{\alpha}_i(x-x_{i+1})+\bar{\beta}_i) & \\ 0 & \end{cases}$$

$$u_h^I(x) = \sum_{i=1}^{n} y_i \varphi_i(x) + \sum_{i=1}^{n} y_i' \bar{\varphi}_i(x)$$

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