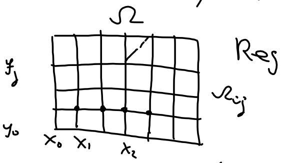
Interpolation in 2D and 3D in $C(\Omega)$

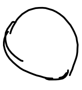
Triangle / quadrilateral meshes. polynomial.

$\{x_i\}_{i=0}^{n}$

Domain (Ω) $\longrightarrow$ Generate a mesh



Regular: Cartesian, polar, spherical



Structured: A fixed pattern between subdomains and nodal points.

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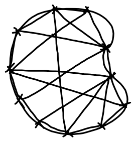
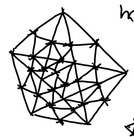
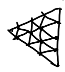


Initial triangulation



middle point rule



An unstructured mesh.

Linear interpolation over a triangle mesh $\{\vec{x}_k\}_{k=1}^{DOF} = \{(x_k, y_k)\}_{k=1}^{DOF}$

<u>Thm.</u> A piecewise linear function in $C(\Omega)$

$$l_1(x) = a_0 + a_1 x + a_2 y$$

is uniquely determined by its values at nodal points

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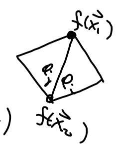
Using the local basis functions, $\varphi_k(\vec{x})$

$$\varphi_k(\vec{x}_j) = \delta_k^j = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{otherwise.} \end{cases}$$

$$u_h^I(x) = \sum_{k=1}^{DOF} f(\vec{x}_k)\,\varphi_k(\vec{x})$$



Is $u_h^I(x)$ continuous? $C(\Omega)$

$$\begin{cases} x = x_1 + a_0 s \\ y = y_1 + b_0 s \end{cases} \qquad \varphi_i(x,y) = \varphi_i(x_1 + a_0 s,\ y_1 + b_0 s)$$

Co-dimension one. One dimensional linear function is uniquely determined by the value at two points.

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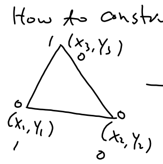
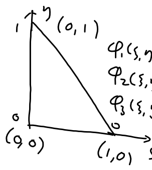
How to construct a basis function



$\xrightarrow{(\xi, \eta)}$

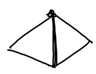


$\varphi_1(\xi,\eta) = \eta$
$\varphi_2(\xi,\eta) = \xi$
$\varphi_3(\xi,\eta) = 1 - \xi - \eta$

$$\begin{cases} \xi = \dfrac{1}{2A}\left((y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1) \right) \\ \eta = \dfrac{1}{2A}\left[-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1) \right] \end{cases}$$

A is the area of the triangle

$$A = \pm \frac{1}{2} \det \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$



u_ξ, u_η

$u, \underline{\underline{u_x, u_y}}$

3
$+2$
$+2$
$???$

$C^1(\Omega)$: Difficult $\quad P_5(x,y) = \sum_{0 \le i+j \le 5} a_{ij} x^i y^j$

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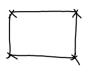
Quadrilaterials

$q(x,y) = a_0 + a_1 x + a_2 y \quad x^2,\ y^2,\ xy$

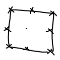


DOF = 4

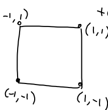
$q(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$, bi-linear.

If we fix one variable, then the other variable is linear.

bi-quadratic $\quad$ DOF = 9



$$q_2(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2 + a_{21}x^2y + a_{12}xy^2 + a_{22}x^2y^2$$



Bi-linear, 4 basis function

$\varphi_1(x,y) = (1-x)(1-y)\,C$

$\varphi_1(-1,-1) = 2 \cdot 2 \cdot C = 1 \quad C = 1/4$

$\varphi_2(x,y) = \bar{C}(x+1)(1-y) = \frac{1}{4}(x+1)(1-y)$

$\varphi_3(x,y) = \frac{1}{4}(1+x)(1+y)$

$\varphi_4(x,y) = \frac{1}{4}(1-x)(1+y)$

$\varphi_i(x,y) = (1 \pm x)(1 \pm y)/4$

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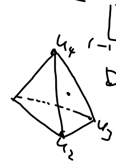
Interpolation $\quad$ One rectangle

$$u(x,y) = \sum_{i=1}^{4} u(x_i, y_i)\, \frac{(1 + x_i x)(1 + y_i y)}{4}$$

3D: Use tetrahedron

$$u(x,y,z) = a + bx + cy + dz$$



DOF = 4

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Numerical Integration → quadrature

Newton - Cotes formula
- left / right Box
- mid-point
- trapezoidal
- Simpson rule

piecewise → composite

Richardson extrapolation

Singular integration → Gaussian quadrature

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Good operation versus 'differentiation'

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Problem: Evaluate

$\int_a^b f(x)\,dx = F(b) - F(a)$ if $F'(x) = f(x)$

in terms of the function values.

$f(x)$ is known.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} \alpha_i f(x_i) = I_n(f)$$

<u>Motivations</u>

$\int e^{x^2} \sin x^2\,dx$

Difficult to find the anti-derivative $F(x)$

$\int_a^b x^n \sin 5x\,dx$ At least n-1 integration by parts

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