Q: 1. What is a spline?

2. What are common spline BC's?

I. $S_3''(x_0)=0$, $S''(x_n)=0$. II $S_3'(x_0)$ and $S_3'(x_n)$ are given.

III periodic

properties of Spline interpolation

'Best approximation'

'least L^2-norm

Localized splines: B-splines

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Precisely: A piecewise cubic function over $\{x_i\}_{i=0}^{n}$ in $C^2(x_0, x_n)$

knots

independent of interpolation.

DOF = $n+3$

In general, any piecewise polynomials over $\{x_i\}_{i=0}^{n}$

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Applications: Matlab spline Toobox

<u>Nurbs</u> Computer Aid design software

Non-uniform rational B-splines $\dfrac{\Sigma \text{ spline}}{\Sigma \text{ spli}}$

Used for solving PDEs.

$$S_3''(x) = M_{j-1}\frac{x-x_j}{x_{j-1}-x_j} + M_j\frac{x-x_{j-1}}{x_j-x_{j-1}} \qquad x_{j-1}\le x\le x_j$$

...

$$S_3(x) = M_{j-1}\frac{(x_j-x)^3}{6h_j} + \frac{M_j(x-x_{j-1})^3}{6h_j} + \frac{y_j-y_{j-1}}{h_j} - \frac{h_j}{6}(M_j-M_{j-1}) \qquad x_{j-1}\le x\le x_j$$

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The linear system of Eqn's for M_j

$$\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = d_j = \frac{6}{h_j+h_{j+1}}\left(f[x_j, x_{j+1}] - f[x_{j-1}, x_j]\right)$$

$j = 1, 2, \ldots, n-1$.

We have $n-1$ eqns. $n+1$ unknowns. We Need 2 more condition.

Natural spline $S_3''(x_0)=M_0=0$, $S_3''(x_n)=M_n=0$

$$\begin{bmatrix} 2 & \lambda_1 & & 0 \\ \mu_2 & 2 & \lambda_2 & \\ & \ddots & \ddots & \lambda_{n-1} \\ 0 & & \mu_{n-1} & 2 \end{bmatrix}\begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1}\end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1}\end{bmatrix}$$

$0<\lambda_j = \dfrac{h_{j+1}}{h_{j+1}+h_j}<1$, $0<\mu_j = \dfrac{h_j}{h_j+h_{j+1}}<1$ harmonic average. over h_j and h_{j+1}

$\lambda_j+\mu_j = 1 < 2 =$ diagonal

The coeffi. matrix is strictly row diagonally dominant. so it is invertible!

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special case $h_j = h_{j-1} = \cdots = h$

$\lambda_j = \dfrac{h}{h+h} = \dfrac{1}{2}$, $\mu_j = \dfrac{1}{2}$.
$$\begin{bmatrix} 2 & \frac12 & & \\ \frac12 & 2 & \frac12 & 0 \\ & \ddots & \ddots & \ddots \\ 0 & & \frac12 & 2 \end{bmatrix}$$

symmetric positive definite. S.P.D.

$S'(x_0)=y_0'$. $S'(x_n)=y_n'$. Go one step back

Recall
$$S_3'(x) = -M_{j-1}\frac{(x_j-x)^2}{2h_j} + M_j\frac{(x-x_{j-1})^2}{2h_j} + f[x_{j-1}, x_j] - \frac{h_j}{6}(M_j-M_{j-1})$$

Take $j=1$ and $x=x_0$.

$$y_0' = -M_0\frac{h_1}{2} + \frac{y_1-y_0}{h} - \frac{h_1}{6}(M_1-M_0)$$

$$-\frac{h_1}{3}M_0 - \frac{h_1}{6}M_1 = y_0' - \frac{y_1-y_0}{h_1}$$

$$\to \quad 2M_0 + M_1 = \frac{6}{h_1}\left(\frac{y_1-y_0}{h_1} - y_0'\right)$$

$(n+1)$ eqns. and $(n+1)$ unknowns

$$\begin{bmatrix} 2 & 1 & \\ \mu_1 & 2 & \lambda_1 \\ & & \end{bmatrix}$$

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periodic $M_0 = M_n$ $y_0 = y_n$

$M_1, M_2, \ldots M_n$. ✓

or $M_0, M_1, \ldots, M_{n-1}$ ✓
$$\begin{bmatrix} * & \times & & \otimes \\ \times & \times & \times & 0 \\ & \ddots & \ddots & \ddots \\ \otimes & & \times & \times \end{bmatrix}$$

$O(n)$

Properties of the Spline interpolations.

(i) Existance and uniqueness for three BC's.

(ii) minimum property (smoothness)

If $f(x)\in C^2[a,b]$. $S_{3,f}$ is the cubic spline interpolation in $C^2[x_0, x_n]$. $S_{3,f}(x_i) = f(x_i)$ $i=0,1,\ldots,n$

then $$\int_a^b |S_{3,f}''|^2 dx \le \int_a^b |f''(x)|^2 dx$$

$$\|S_{3,f}''\|_{L^2(x_0,x_n)} \le \|f''\|_{L^2(x_0,x_n)};$$

h^2 h^3

(iii) Best approximation property ✓

$x_0=a$, $x_n=b$. $$\int_a^b |f''-S_{3,f}''|^2 dx \le \int_a^b |f''-S_3''|^2 dx \qquad \|S_{3,f}(x)-f(x)\|_\infty \le Ch^4$$

Where $S_3(x)$ is a piecewise cubic function in $C^2[x_0, x_n]$ and satisfies the same BC.

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Proof: Use natural BC, $S''(x_0)=0$, $S''(x_n)=0$.

$$\int_a^b (f''-S_*'')^2 dx = \int_a^b \left((f'')^2 - 2f''S_*'' + (S_*'')^2\right) dx \qquad S_* = S_{3,f}$$

$$= \int_a^b \left((f'')^2 - 2\underbrace{(f''-S_*'')S_*''}_{=0} - (S_*'')^2 \right) dx \geq 0$$

If $\int_a^b (f''-S_*'')S_*''\,dx = 0$, then we get

$$\int_a^b (f'')^2 dx \geq \int_a^b (S_*'')^2 dx$$

$$\int_a^b \underline{(f''-S_*'')}\, S_*''\, dx = (f'-S_*')S_*''\Big|_a^b - \int_a^b (f'-S_*')S'''\,dx$$

$$= (f'-S_*')S''\Big|_a^b - (f-S_*)S'''\Big|_a^b + \int_a^b (f-S)S^{(4)}(x)\,dx$$

$=0$ $\quad$ $=0$

Natural spline $S''(a) = S''(b) = 0$

Clamped spline $f'(a) - S_*'(a) = 0$ and $f'(b) - S_*'(b) = 0$

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The "best approx."

$$\int_a^b (f''-S_3'')^2 dx = \int_a^b (f''-S_3''+S_*''-S_*'')^2 dx$$

$$= \int_a^b \underbrace{(f''-S_*'')^2}_{\geq 0} - 2\underbrace{(S_3''-S_*'')(f''-S_*'')}_{=0} + \underbrace{(S_3''-S_*'')^2}_{\geq 0} dx \geq \int_a^b (f''-S_*'')^2 dx$$

$$\geq \int_a^b (f''-S_*'')^2 dx$$

$5+3 \geq 5$

$$\int_a^b (S_3''-S_*'')(f''-S_*'')dx$$

$$= \underbrace{(f'-S_*')\Big|_a^b (S_3''-S_*'')\Big|_a^b}_{=0} - \int_a^b \underbrace{(f'-S_*')(S_3'''-S_*''')}_{=0} dx$$

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