Q. What n should we take so that the composite Trapezoidal / Simpson can have 6 significant digits ?

$f''(\eta) \sim O(1)$

T: $E_n \sim O(h^2)$, $h = \frac{b-a}{n}$ $\quad |E_n| \le 10^{-6}$

S: $E_n \sim O(h^4)$

$\frac{1}{h^2} \le 10^{-6}$

$n \ge 1000$

$\frac{1}{h^4} \le 10^{-6}$ $\quad n \ge 10^{6/4} \approx 32$

No. of function evaluation $O(n)$ Trap.

$O(2n)$ Simpson's

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Relation between composite trapezoidal and Simpson's rule

$$S_n = \frac{4T_{2n} - T_n}{4-1}$$

Richardson extrapolation

A useful acceleration technique.

Idea: Given $A(h) = a_0 + a_1 h + a_2 h^2 + \cdots \quad (1)$

$\lim_{h \to 0} A(h) = a_0$

$A(h/2) = a_0 + a_1 \frac{h}{2} + a_2 \left(\frac{h}{2}\right)^2 + \cdots \quad (2)$

$A(\frac{h}{2}) - \frac{1}{2}A(h) = a_0 - \frac{1}{2}a_0 + 0 + a_2\left(\left(\frac{h}{2}\right)^2 - \frac{h^2}{2}\right)$

$$a_0 = \frac{A(h/2) - \frac{1}{2}A(h)}{1 - \frac{1}{2}} \pm a_2 \frac{3h^2}{4} + O(h^3)$$

Richardson extrapolation.

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$T_n(h) = a_0 + a_2 h^2 + a_4 h^4 + \cdots$ composite trapezoidal

$T_n(\delta h) = a_0 + a_2(\delta h)^2 + a_4(\delta h)^4 + \cdots$

$T_n(\delta h) - \delta^2 T_n = a_0 - \delta^2 a_0 + a_4((\delta h)^4 - \delta^2 h^4) + O(h^6)$

Solve for a_0

$$a_0 = \frac{T_n(\delta h) - \delta^2 T_n}{1 - \delta^2} + O(h^4)$$

If we take $\delta = \frac{1}{2}$. we get

$$a_0 = \frac{T_n(\frac{h}{2}) - \frac{1}{4}T_h}{1 - \frac{1}{4}} = \frac{4T_n(h/2) - T_n(h)}{3}$$

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Richardson extrapolation table.

If $\underset{O(h)}{A(h)} = a_0 + a_1 h + \underset{O(h^2)}{a_2 h^2} + \cdots \quad a_1 \ne 0$

$h \quad A_{0,0} = A(h)$

$h/2 \quad A_{1,0} = A(\frac{h}{2}) \quad A_{1,1} = \frac{A_{1,0} - \frac{1}{2}A_{0,0}}{1 - \frac{1}{2}}$

$h/4 \quad A_{2,0} = A(\frac{h}{4}) \quad A_{2,1} = \frac{A_{2,0} - \frac{1}{2}A_{1,0}}{1 - \frac{1}{2}} \quad A_{22} = \frac{A_{2,1} - (\frac{1}{2})^2 A_{1,1}}{1 - (\frac{1}{2})^2}$

$h/8 \quad A_{3,0} = A(\frac{h}{8}) \quad A_{3,1} = \frac{A_{3,0} - \frac{1}{2}A_{2,0}}{1 - \frac{1}{2}} \quad A_{32} = \frac{A_{3,1} - (\frac{1}{2})^2 A_{2,1}}{1 - (\frac{1}{2})^2}$

$$A_{33} = \frac{A_{32} - (\frac{1}{2})^3 A_{22}}{1 - (\frac{1}{2})^3}$$

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Ex. Appy the Richardson extrapolation to a numerical differenation

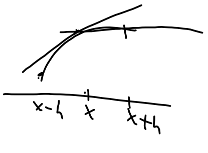
$f'(x) \approx \frac{f(x+h) - f(x)}{h}$ $\quad e^x \sin x \cos x$

$E = \left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| \le O(h)$

$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$

$E = \left| f'(x) - \frac{f(x+h) - f(x-h)}{2h} \right|$

$= \frac{|f''(x)|}{2} h^2 + O(h^4)$ $\quad f''$



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Romberg Integration. Apply the Richardson extrapolation to the composite rule

The Richarson table.

$O(h^2) \quad O(h^4) \quad O(h^6)$

$T_h \to A_{0,0}$

$T_{h/2} \to A_{1,0} \quad A_{11}$ $\qquad A_{i,1} = \frac{2^2 A_{i,0} - A_{i-1,0}}{2^2 - 1}$

$T_{h/2^2} \to A_{2,0} \quad A_{21} \quad A_{22}$ $\qquad A_{i,2} = \frac{2^4 A_{i,1} - A_{i-1,1}}{2^4 - 1}$

$A_{n,0} \quad A_{n1} \quad A_{n2} \quad \cdots \quad A_{nn}$

$$A_{i,k+1} = \frac{2^{2(k+1)} A_{i,k} - A_{i-1,k}}{2^{2(k+1)} - 1}$$

Why don't we have h^{2k+1} terms in the error

Thm: If $f(x) \in C^{2k+2}([a,b])$ for any $k > 0$. them

$$R_n = \int_a^b f(x)\,dx - T_n$$

$$= -\sum_{i=1}^{n} \frac{B_{2i}}{(2i)!} h^{2i} \left(f^{(2i-1)}(b) - f^{(2i-1)}(a) \right) - \frac{B_{2n+2}}{(2n+2)!} h^{2n+2} (b-a) f^{(2n+2)}(\eta)$$

B_{2i} are the Bernoulli numbers, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$

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Given a $\varepsilon > 0$, $\varepsilon = 10^{-6}$, how do we choose n?
How do we know error without the true $A_{n,n}$ integration. Auto integration: posterior error estimate.

$I(f) = \int_a^b f(x)\,dx$, then

$$I(f) - A_{i,k} = \frac{1}{4^k}\left(I(f) - A_{i-1,k}\right) \qquad k=1,\ O(h^2)$$

$$A_{i,k} = I(f) + C h^{2k+2}$$

$$A_{i+1,k} = I(f) + c\,\frac{h^{2k+2}}{2^{2k+2}}$$

$$\frac{A_{i,k} - I(f)}{A_{i+1,k} - I(f)} \approx 4^{k+1}$$

$$\Rightarrow \quad A_{i,k} - I(f) = 4^{k+1}\left(A_{i+1,k} - I(f)\right)$$

Solve for $I(f)$, we get

$$I(f) \approx \frac{4^{k+1} A_{i+1,k} - A_{i,k}}{4^{k+1} - 1}$$

Richardson extrapolation

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