Q: For Newton-Cotes quadrature formula, should we take n even ✓ / odd ?

Knowledge base: Mean-Value Theorem for

$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$ if $g(x) \geq 0$ / ≤ 0 ?

Material Today:

Theory: Error analysis of N-C.

Concept: Algebraic precision.

Algorithm: composite quadrature. mostly used.

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Error Analysis for Newton-Cotes formula.

Recall: $f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)$

$$\left|\int_a^b f(x)dx - \int_a^b p_n(x)dx\right| \leq \int_a^b \frac{|f^{(n+1)}(\xi)|}{(n+1)!}|\omega_{n+1}(x)|dx$$

$\int_a^b p_n(x)dx = \sum_{i=0}^n \alpha_i f(x_i)$

If $f(x) \in C^{n+1}(a,b)$

$$\leq \frac{\max_{a\leq x\leq b}|f^{(n+1)}(x)|}{(n+1)!}\int_a^b |\omega_{n+1}(x)|dx$$

Equally spaced

$x_i = a + ih$, $x = a + th$, $dx = h\,dt$; $x - x_i = a+th-(a+ih)$

$$|E_n(f)| \leq \begin{cases} h^{n+2}\frac{f^{(n+1)}_{max}}{(n+1)!}\int_0^n |t(t-1)\cdots(t-n)|dt, & \text{closed} \\ h^{n+1}\frac{f^{(n+1)}_{max}}{(n+1)!}\int_{-1}^{n+1} |t(t-1)\cdots(t-n)|dt, & \text{open}\end{cases}$$

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$f(x) = \sum_{j=0}^k a_j x^j$, $0 \leq k \leq n$, $E_n(f) \equiv 0$.

Corollary. If $f(x)$ is a polynomial of degree $k \leq n$, then the Newton-Cotes formula is exact!

We can get better results if we take $n = 2k$, $k = 0, 1, \ldots$

Ex: The midle point rule (open, $n=0$). It will be exact for any linear function $f(x) = \alpha + \beta x$,

Exact integration $\int_a^b f(x)dx = \int_a^b (\alpha + \beta x)dx$
$= \alpha(b-a) + \frac{\beta}{2}(b-a)(b+a)$
$= (b-a)(\alpha + \beta\frac{a+b}{2})$

Mid-point rule $\int_a^b f(x)dx = (b-a)f(\frac{a+b}{2})$
$= (b-a)(\alpha + \beta\cdot\frac{a+b}{2})$ $\equiv$

<u>Thm:</u>

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)\right| \leq \frac{\max_{a\leq x\leq b}|f''(x)|}{24}(b-a)^3$$ ✓

For trapezoidal rule ($n=1$, closed)

$$|E_1(f)| \leq \frac{\max_{a\leq x\leq b}|f''(x)|}{12}(b-a)^3$$

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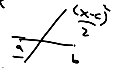
Proof: $f(x) = f(c + x - c)$, $c = \frac{a+b}{2}$, Taylor expansion

$= f(c) + f'(c)(x-c) + \frac{f''(\xi)}{2}(x-c)^2$ ✓

$\frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3$ Generalized Taylor expansion

$\int_a^b f(x)dx = f(c)(b-a) + f'(c)\cdot 0 + \int_a^b \frac{f''(\xi(x))}{2}(x-c)^2 dx$

$\int_a^b (x-c)dx$



$= f(c)(b-a) + \frac{f''(\eta)}{2}\int_a^b (x-c)^2 dx$

mid-point

error term.

$= f(c)(b-a) + \frac{f''(\eta)}{2}\left.\frac{(x-c)^3}{3}\right|_a^b$

$= f(c)(b-a) + f''(\eta)\frac{(b-\frac{b+a}{2})^3}{3}$

$= f(c)(b-a) + \frac{f''(\eta)}{24}(b-a)^3$ ✓

$b - c = c - a$

$\left(\frac{b-a}{2}\right)^3 = \frac{(b-a)^3}{8}$

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More accurate error estimate for Newton-Cotes formula.

$E_n(f) = \int_a^b f(x)dx - \int_a^b p_n(x)dx$ $\sim \sum_{i=0}^n \alpha_i f(x_i)$

f(x)

X

$$= \begin{cases} \frac{K_n}{(n+1)!}h^{n+2}f^{(n+1)}(\xi) & \text{if } n = 2k+1,\ k = 0, 1, \ldots \\ \frac{M_n}{(n+2)!}h^{n+3}f^{(n+2)}(\xi) & \text{if } n = 2k,\ k = 0, 1, \ldots \end{cases}$$

$M_n = \int_\alpha^\beta t^2(t-1)(t-2)\cdots(t-n)\,dt$ $\quad \alpha = 0, \beta = n$ closed

$K_n = \int_\alpha^\beta t(t-1)\cdots(t-n)\,dt$. $\quad \alpha = -1, \beta = n+1$.

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The (degree of) algebraic precision (exactness) of a quadrature formula $\int_a^b f(x)dx \approx \sum \alpha_i f(x_i)$

If $E(p_k) = \int_a^b p_k(x)dx - \sum \alpha_i f(x_i) = 0$ for $k = 0, 1, \ldots m$, where $p_k(x)$ is <u>any</u> polynomial of degree k in any interval (a,b) but $E(p_{m+1}(x)) \neq 0$ for some polynomial of degree $(m+1)$, then the quadrature formula has the algebraic precision m.

<u>Ex:</u> What's the algebraic precision for the middle point rule. ≥ 1 $= 1$

$\int_a^b x^2 dx = \frac{b^3-a^3}{3} = \frac{(b-a)}{3}(a^2+ab+b^2)$

$\int_a^b x^2 dx = (b-a)\left(\frac{a+b}{2}\right)^2 = \frac{(b-a)}{2}\left(\frac{a^2+2ab+b^2}{2}\right)$

$a = 0$, $b = 1$

Trap. $= 1$.

$\int_a^b p_n(x)dx = \sum_{i=0}^n \alpha_i f(x_i)$

What's the highest algebraic precision can we have?

n ✗

$n+1$ ✓

$n+2$

$2n+1$ ✓ (Gaussian Quad. α_i, x_i

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Algorithm. → piecewise → composite (compound)

Purpose: Increase accuracy.

Overcom Runge's phen.

Recursive relation

decrease the requirements of regularities.

degree of smoothness of high order differentiability

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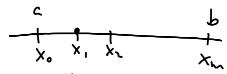
Composite trapezoidal formula. f'' $C^{1+\varepsilon}$, $\varepsilon>0$

$[a, b]$



$x_i = a + ih$, $h = \frac{(b-a)}{m}$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \approx \sum_{i=0}^{m-1} \frac{x_{i+1}-x_i}{2}\left[f(x_i)+f(x_{i+1})\right]$$

$$= \frac{h}{2} \sum_{i=0}^{m-1} \left[f(x_i)+f(x_{i+1})\right]$$

$$= \frac{h}{2}\left[f(a)+f(b)\right] + \sum_{i=1}^{m-1} f(x_i)\,h.$$

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