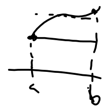
Q: Which quadrature is the best?

A: left/right box rule one f



B: Mid-point rule one f ✓ open

C: Trapezoidal rule 2 f's

D: Simpson's rule 3 f's

Open only use interior pts.

closed Need to use end points

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Newton-Cotes quadrature closed open

and analysis

$f(x) \to p_n(x)$

$$\int_a^b f(x)\,dx \approx \int_a^b p_n(x)\,dx = \sum_{i=0}^{n} \omega_i f(x_i)$$

weights

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Applications. Area, volume, center

Finite element methods $\int x$

ODE/PDE → mesh → Integral forms.

$AU = F$

$A = \{a_{ij}\}$ $\quad a_{ij} = \int_{T_i} \varphi_i'(x)\varphi_j'(x)\,dx$ stiffness

$\varphi_i(x_j) = \delta_i^j$. hat functions → need open formula

$-u'' + qu = f$ $\quad 0<x<1$

$u(0)=0,\ u(1)=0$ $\quad \int_{T_i} q(x)\varphi_i(x)\varphi_i(x)\,dx$

Integral equations,

$0<\alpha<1$

fractional derivative $\quad \dfrac{d^\alpha u}{dt^\alpha} = \dfrac{1}{\Gamma(\,)}\displaystyle\int_0^{x} \dfrac{\frac{d}{dx}u}{(x-t)^\alpha}$

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General formula and error estimates

$f(x) \to p_n(x)$, polynomial interpolation

$$\int_a^b f(x)\,dx \approx \int_a^b p_n(x)\,dx = \int_a^b \sum_{i=0}^{n} l_i(x) f(x_i)$$

$$= \sum_{i=0}^{n}\left(\int_a^b l_i(x)\,dx\right) f(x_i)$$

$$= \sum_{i=0}^{n} \omega_i f(x_i).$$

The weights $\quad \omega_i = \int_a^b l_i(x)\,dx$

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If we use equally spaced nodes, the formula then is called the Newton-Cotes formula. $\quad x_i - x_{i-1} = h,$

$x_i = x_0 + ih, \quad i = 0,1,\dots,n. \quad h = \frac{b-a}{n}$

$$W_i = \int_a^b l_i(x)\,dx = \int_a^b \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}\,dx$$

Let $x = x_0 + th$ $\quad x - x_j = x_0 + th - (x_0 + jh)$

$dx = h\,dt$ $\quad = (t-j)h \quad 0 \le t \le n$

$x_i - x_j = x_0 + ih - (x_0 + jh)$

$= (i-j)h$

$$\omega_i = \int_0^n h \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{t-j}{i-j}\,dt$$

$\int_a^b f(x)\,dx = b-a$

$I_n(f) = h\sum_{i=0}^{n} \bar{W}_i f(x_i)$. $\quad h\sum_{i=0}^{n}\bar{W}_i = b-a$ Consistency condition.

$\bar{W}_i$ are called the coefficients of the Newton-Cotes formula.

$$\bar{W}_i = \int_0^n \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{t-j}{i-j}\,dt$$

Denominator

$i(i-1)\cdots 1 = i!$

$1\ 2 \cdots (n-i) = (n-i)!$

$(-1)(-1)\cdots(-1)$

$$= \frac{(-1)^{n-i}}{i!\,(n-i)!}\int_0^n \prod_{\substack{j=0\\ j\neq i}}^{n} (t-j)\,dt$$

Ex: $n=1,\ h = b-a$

$$\bar{W}_0 = \frac{(-1)}{1}\int_0^1 (t-1)\,dt = -\frac{(t-1)^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\bar{W}_1 = \int_0^1 \frac{(t-0)}{(1-0)}\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_a^b f(x)\,dx = (b-a)\left(\tfrac{1}{2}f(a) + \tfrac{1}{2}f(b)\right)$$

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Ex 2: $n=2$, $h = \frac{b-a}{2}$

$$\bar{W}_0 = \int_0^2 \frac{(t-1)(t-2)}{(0-1)(0-2)}\,dt = \frac{1}{2}\int_0^2 (t^2-3t+2)\,dt = \frac{1}{3}$$

$$\bar{W}_1 = \int_0^2 \frac{(t-0)(t-2)}{(1-0)(1-2)}\,dt = \frac{4}{3}$$

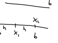
$$W_2 = \int_0^2 \frac{(t-0)(t-1)}{(2-0)(2-1)}\,dt = \frac{1}{3}$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left(\frac{1}{3}f(a) + \frac{4}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{3}f(b)\right)$$

$$= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

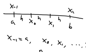
1. Computer language → Fortran ✓, C, C++, Python
2. Visualization Tool Matlab. Tec plots.
3. Symbolic package < Maple, Mathematica

Open formulas.



$h = \frac{b-a}{2}$

$n=1$



$h = \frac{b-a}{3}$

Generally. $x_{-1} = a,\ x_0, x_1, \dots, x_n,\ x_{n+1} = b,\ h = \frac{b-a}{n+2}$

$$\int_a^b f(x)\,dx = h\sum_{i=0}^{n} W_i f(x_i)$$

$$W_i = \int_{-1}^{n+1} \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(t-j)}{(i-j)}\,dt = \frac{(-1)^{n-i}}{i!\,(n-i)!}\int_{-1}^{n+1}\prod_{\substack{j=0\\ j\neq i}}^{n}(t-j)\,dt$$

$n=0$. $\quad W_0 = \int_{-1}^{1} dt = 2.$

$$\int_a^b f(x)\,dx = hW_0 f(x_0) = \frac{b-a}{2}\cdot 2\, f\left(\frac{a+b}{2}\right) = (b-a)f\left(\frac{a+b}{2}\right)$$

$n=1$: $\quad x_{-1} = a,\ h = \frac{b-a}{3},\ x_0 = a + \frac{b-a}{3}$

$x_1 = a + \frac{2(b-a)}{3}$

$$\omega_0 = \int_{-1}^{2} \frac{(t-1)}{(0-1)}\,dt = -\frac{(t-1)^2}{2}\Big|_{-1}^{2} = \frac{3}{2}$$

$$\omega_1 = \int_{-1}^{2}\left(\frac{t}{1-0}\right)dt = \frac{3}{2}$$

$$\int_a^b f(x)\,dx = \frac{b-a}{3}\left[\frac{3}{2}f\left(a+\frac{b-a}{3}\right) + \frac{3}{2}f\left(a+\frac{2(b-a)}{3}\right)\right)$$

A new quadrature formula

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Stability and error analysis of Newton-Cotes formulas.

Consistency. $\int_a^b f(x)\,dx \approx h \sum_{i=0}^{n} w_i f(x_i)$

$$h \sum_{i=0}^{n} w_i = (b-a)$$

Thm: If $w_i \geq 0$, then the N-C formula is stable.

If $f(x_i) = f(x_i) + \varepsilon_i$ $\quad |\varepsilon_i| \leq \varepsilon$

$$\left| h \sum_{i=0}^{n} w_i f(x_i) - h \sum_{i=0}^{n} w_i \left(f(x_i) + \varepsilon_i\right) \right|$$

$$= h \left| \sum_{i=0}^{n} w_i \varepsilon_i \right| \overset{w_i \geq 0}{\leq} h \sum_{i=0}^{n} w_i |\varepsilon_i|$$

$$\leq h \sum_{i=0}^{n} w_i \varepsilon = \varepsilon (b-a) \quad \checkmark \text{ stable.}$$

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