MA 587
Homework #4
Due

1. Which of the following problems have a unique solution? Why? (explain the reason(s), but no proof is needed). Or give condition that the solution exist and/or unique.

$$A: \begin{cases} -u'' = f, & 0 < x < 1, \\ u(0) = 0, & u(1) = 0; \end{cases} \qquad B: \begin{cases} -u'' + u = f, & 0 < x < 1, \\ u'(0) = 0, & u'(1) = 0; \end{cases} \qquad C: \begin{cases} -u'' + q(x)u = f, & 0 < x < 1, \\ u'(0) = 0, & u'(1) = 0; \end{cases}$$

where $q(x) \geq q_{min} > 0$. What happens if we relax the condition to $q(x) \geq 0$? **Hint: Use the Lax-Milgram Lemma**.

2. Given

$$u''''(x) + q(x)u = f, \quad 0 < x < 1,$$

$$u(0) = u'(0) = 0, \quad u(1) = 0, \quad \alpha u''(1) + \beta u'(1) = \gamma$$

   (a) Derive the weak form.
   (b) Give sufficient conditions for $q(x)$, α, and β such that the weak form has a unique solution.
   (c) Find an upper bound for the solution.

3. Derive the week form for the PDE

$$-\left(a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy}\right) + b_1 u_x + qu = f(x,y), \quad (x,y) \in \Omega, \quad u(x,y)\Big|_{\partial\Omega} = 0,$$

assume that all the coefficients are function of (x, y). What happens if the BC is part of Dirichlet $u|_{\partial\Omega_1} = 0$ and part of Nuemann $u_n|_{\partial\Omega_2} = g(x,y)$?

4. (An eigenvalue problem, optional.) Consider

$$-(pu')' + qu - \lambda u = 0\,, \quad 0 < x < \pi\,, \tag{1}$$

$$u(0) = 0, \quad u(\pi) = 0\,. \tag{2}$$

   (a) Find the weak form of the problem and check whether the conditions of the Lax-Milgram Lemma are satisfied.
   (b) Use the 1D FE package with linear basis functions and a uniform grid to solve the eigenvalue problem

$$-(pu')' + qu - \lambda u = 0\,, \quad 0 < x < \pi\,,$$

$$u(0) = 0\,,\ u'(\pi) + \alpha u(\pi) = 0\,,$$

$$\text{where} \quad p(x) \geq p_{min} > 0\,,\ q(x) \geq 0\,,\ \alpha \geq 0\,.$$

   in each of the following two cases:

      i. $p(x) = 1$, $q(x) = 1$, $\alpha = 1$.

ii. $p(x) = 1 + x^2$, $q(x) = x$, $\alpha = 3$.

Try to solve the eigenvalue problem with $M = 5$ and $M = 20$. Print out the eigenvalues but not the eigenfunctions. Plot all the eigenfunctions in a single plot for $M = 5$, and plot two typical eigenfunctions for $M = 20$ (6 plots in total).

Hint: The approximate eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_M$ and the eigenfunction $u_{\lambda_i}(x)$ are the generalized eigenvalues of

$$Ax = \lambda B x \,,$$

where A is the stiffness matrix and $B = \{b_{ij}\}$ with $b_{ij} = \int_0^\pi \phi_i(x)\phi_j(x)dx$. You can generate the matrix B either numerically or analytically; and in Matlab you can use $[V, D] = EIG(A, B)$ to find the generalized eigenvalues and the corresponding eigenvectors. For a computed eigenvalue λ_i, the corresponding eigenfunction is

$$u_{\lambda_i}(x) = \sum_{j=1}^{M} \alpha_{i,j}\phi_j(x) \,,$$

where $[\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,M}]^T$ is the eigenvector corresponding to the generalized eigenvalue.

Note: if we can find the eigenvalues and corresponding eigenfunctions, the solution to the differential equation can be expanded in terms of the eigenfunctions, similar to Fourier series.

5. Consider the Poisson equation

$$\begin{aligned} -\Delta u(x,y) &= 1, \quad (x,y) \in \Omega \\ u(x,y)|_{\partial\Omega} &= 0, \end{aligned}$$

where Ω is the unit square. Using a uniform triangulation, derive the stiffness matrix and the load vector for $N = 2$, *i.e.* $h = 1/3$:

(a) The nodal points are ordered as $(1/3, 1/3)$, $(2/3, 1/3)$; $(1/3, 2/3)$, and $(2/3, 2/3)$.

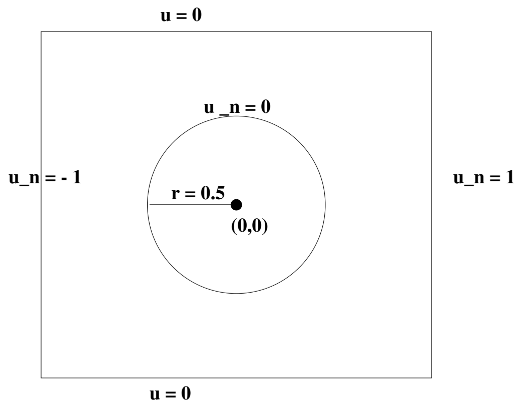
(b) The nodal points are ordered as $((1/3, 2/3)$, $(2/3, 1/3)$; $(1/3, 1/3)$, and $(2/3, 2/3)$.

Write down the basis function centered at $(1/3, 2/3)$ explicitly. You can use the formula from the book directly.

6. Use Matlab PDE toolbox to solve the following parabolic equation for $u(x, y, t)$ and make relevant plots:

$$\begin{aligned} u_t &= u_{xx} + u_{yy}, \quad (x,y) \in [-1 \;\; 1] \times [-1 \;\; 1] \\ u(x,y,0) &= 0 \end{aligned}$$

The geometry and the boundary conditions are defined in Figure **??**. Show a couple of plots of the solution (mesh, contour etc.). Find the global extrems of $u(x, y)$ using the computed solution. Are they attained at the boundary?



7. Down-load the Matlab source code *f.m, my_assemb.m, uexact.m* from Moodle or the class web-page. Use the exported mesh of the geometry of the **third** problem of the Lab practice from Matlab to solve the Poisson equation

$$-(u_{xx} + u_{yy}) = f(x, y).$$

The Dirichlet boundary condition is determined from the exact solution

$$u(x, y) = \frac{1}{4}\left(x^2 + y^4\right)\sin \pi x \cos 4\pi y.$$

Find $f(x, y)$. Plot the domain and mesh. Find the total of degree of the freedom. Plot the solution and the error.

Extra credit: Compute the errors in $\|E\|_0$, i.e. the L^2, $\|E\|_a$, i.e. the energy norm; and $\|E\|_1$, i.e. the H^1 norm. What can you expect the convergence order for all the norms?