MA 587
Homework #3
Due Date: March 31, 2022

1. Consider the Sturm-Liouville problem

$$\begin{aligned} -((1+x^2)u')' + xu &= f, \quad 0 < x < 1, \\ & u(1) = 2. \end{aligned}$$

Transform the problem to a problem with homogeneous Dirichlet boundary condition at $x = 1$, that is, $\hat{u} = u(x) - u_0(x)$, and find $u_0(x)$ with $u_0(1) = 2$, *c.f.* the text book. Write down the weak form for each of the following case: (a), $u(0) = 3$; (b), $u'(0) = 3$; (c), $u(0) + u'(0) = 3$.

Note: Three sub-problems.

2. (Purpose: Review abstract finite element methods.) Consider the Sturm-Liouville problem

$$\begin{aligned} -u'' + u &= f, \quad 0 < x < \pi, \\ u(0) = 0, & \qquad u(\pi) + u'(\pi) = 1. \end{aligned}$$

Let V_f be the finite dimensional space

$$V_f = span\,\{\,x,\ \sin(x), \sin(2x)\,\}\,.$$

Find the best approximation to the solution of the weak form from V_f in the energy norm. You can use either analytic derivation or computer software packages, for example, Maple, Matlab, SAS, etc. to solve the problem. Take $f = 1$ for the computation. Compare with this approach with the finite element method using the hat basis functions.

3. Consider the Sturm-Liouville problem

$$\begin{aligned} -(pu')' + qu &= f, \quad a < x < b, \\ u(a) = 0, & \qquad u(b) = 0. \end{aligned}$$

Given a triangulation $a = x_0 < x_1 \cdots < x_M = b$ and a finite element space

$$V_h = \Big\{\ v(x),\ v(x) \quad \text{is piecewise cubic function over the triangulation,} \quad v(x) \in H_0^1(a,b)\Big\}$$

   (a) Find the dimension of V_h.
   (b) Find all non-zero shape functions $\psi_i(\xi)$, $-1 \le \xi \le 1$ and plot them.
   (c) What is the size of the local stiffness matrix and load vector? Sketch of the assembling process.
   (d) List a few advantages and disadvantages of this finite element space compared with the piecewise continuous linear finite dimensional spaces, i.e., the hat functions.
   (e) Compare this $P_3(\Omega_h) \in H_0^1(a,b)$ with $P_3(\Omega_h) \in H_0^2(a,b)$. Explain as much as you can.

4. You need to down-load the files of one dimensional FEM code from the class homepage. Let

$$u(x) = e^x \sin x, \quad p(x) = 1 + x^2, \quad q(x) = e^{-x}, \quad c(x) = 1,$$

and $f(x)$ is determined from the following differential equation

$$-(pu')' + c(x)u' + qu = f, \quad a \le x \le b.$$

Use this example to get familiar with the FEM package by trying the following boundary conditions.

(a) Dirichlet BC at $x = a$ and $x = b$, where $a = -1$, $b = 2$.

(b) Neumann BC at $x = a$ and Dirichlet BC at $x = b$, where $a = -1$, $b = 2$.

(c) Mixed BC, $\gamma = 3u(a) - 5u'(a)$, at $x = a = -1$, and Neumann BC at $x = b = 2$.

Tabulate the errors in the infinity norm

$$e_M = \max_{0 \le i \le M} |u(x_i) - U_i|$$

at the nodes and auxiliary points as the follows

M	Basis	Gaussian	error	e_M/e_{2M}

for different $M = 4, 8, 16, 32, 64$ (nnode=M+1), or the **closest integers** if necessary, with linear. quadratic, and cubic basis functions, **NO** plots please. What are the convergence orders? Note: the method is second, third, or fourth, order convergent if the ratio e_M/e_{2M} approaches 4, 8, 16 respectively.

For the **last case**, (1) print out the stiffness matrix for the *linear basis function* with $M = 5$. Is it symmetric? (2) Plot the computed solution against the exact one and the error plot for the case of the linear basis function. Take enough points to plot the exact solution so that we can see the whole picture. (3) Also plot the error versus $h = 1/M$ in the log-log scale for three different basis. The slope of such a plot is the convergence order of the method employed. For this problem, you will only produce *five* plots for the last case.

5. **Extra Credit: You do not have to do it.** Choose only one. Find the energy norm, H^1 norm, and L^2 norm of the error and do the grid refinement analysis.

6. **Extra Credit: You do not have to do it.**

   (a) Show that the finite element solution is exact at nodal points for $-u''(x) = f(x)$, $a < x < b$, $u(a) = u(b) = 0$ using the $P_1(\Omega_h) \in H_0^1(a, b)$, that is $u_h(x_i) = u(x_i)$.

   (b) Construct a new function

   $$\hat{u}_h(x) = u_h(x) + \frac{f(x)}{2}(x - x_i)(x - x_{i+1}), \qquad x_i \le x \le x_{i+1}$$

   has third order accuracy in $L^2(a, b)$, and second order accuracy in $H^1(a, b)$ assuming that $f(x) \in C^1(a, b)$.