MA 587
Homework #2
**Due March 8**

1. Take $n = 3$, the number of variables, describe the Sobolev space $H^3(\Omega)$, i.e. $m = 3$, in terms of $L^2(\Omega)$ (*list all terms*). Also explain the inner product, the norm, the Schwartz inequality, the distance, and the Sobolev embedding theorem in this space as we did in class (*use multi-index notations*).

2. Apply the Cauchy-Schwartz inequality for the following situations.

   (a) In $R^n$ space. Let $x \in R^n$, $y = [1\ 1\ \cdots, 1]^T \in R^n$, where $x^T$ is the transpose of a vector $x$. Consider $(x, y)_{R^n}$.

   (b) In $L^2(R)$ space, where $R$ is the unit square in two dimensions, that is, $-1 < x, y < 1$. Let $g(x, y) = 1$, $f(x, y) \in L^2(R)$. Consider $(f, g)_{L^2(R)}$.

   (c) The same functions as above but consider $(f, g)_{H^1(R)}$.

3. Consider the function $v(x) = |x|^\alpha$ on $\Omega = (-1, 1)$ with $\alpha \in \mathcal{R}$. For what values of $\alpha$ is $v \in H^0(\Omega)$? (Consider negative $\alpha$ as well). For what values is $v \in H^1(\Omega)$? in $H^m(\Omega)$? For what values of $\alpha$ is $v \in C^m(\Omega)$?

   **Hint:** Generally

$$|x|^\alpha = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ (-x)^\alpha & \text{if } x < 0. \end{cases}$$

   However, if $\alpha = 2k$, where $k$ is a non-negative integer, then

$$|x|^\alpha = \begin{cases} x^{2k} & \text{if } \alpha = 2k, k > 0 \\ 1 & \text{if } \alpha = 0. \end{cases}$$

   Also

$$\lim_{x \to 0} |x|^\alpha = \begin{cases} 0 & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ \infty & \text{if } \alpha < 0, \end{cases} \qquad \int_{-1}^{1} |x|^\alpha dx = \begin{cases} \dfrac{2}{\alpha + 1} & \text{if } \alpha > -1 \\ \infty & \text{if } \alpha \leq -1. \end{cases}$$

4. Are each of the following statements true or false? Justify your answers.

   (a) If $u \in H^2(0, 1)$ then $u'$ and $u''$ are both continuous functions.

   (b) If $u(x, y) \in H^2(\Omega)$, then $u(x, y)$ may not have continuous partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$? Does $u(x, y)$ have first and second order *weak* derivatives? Is $u(x, y)$ continuous in $\Omega$?

5. Consider the Sturm-Liouville problem:

$$\begin{aligned} -\left(p(x)u(x)'\right)' + q(x)u(x) &= f(x), \quad 0 < x < \pi, \\ \alpha u(0) + \beta u'(0) = \gamma, &\qquad u'(\pi) = u_b, \end{aligned}$$

   where

$$0 < p_{min} \leq p(x) \leq p_{max} < \infty, \quad 0 \leq q_{min} \leq q(x) \leq q_{max} < \infty.$$

(a) Derive the weak form for the problem. Define a bilinear form $a(u, v)$ and a linear form $L(v)$ to simplify it. What is the energy norm?

(b) What kind of restrictions should we have for $\alpha$, $\beta$, and $\gamma$ in order that the weak form has a solution?

(c) Determine the space where the solution resides under the weak form.

(d) If we look for a finite element solution in a finite dimensional space $V_h$ using a conforming finite element method, should $V_h$ be a subspace of $C^0$, $C^1$, $C^2$?

(e) Given a triangulation $x_0 = 0 < x_1 < x_2 \cdots < x_{M-1} < x_M = \pi$, if the finite dimensional space is generated by the hat functions, what kind of structure do the local and global stiffness matrix and the load vector have? Is the resulting linear system of equations formed by the global stiffness matrix and the load vector symmetric, positive definite, and banded?

(f) Let

$$p(x) = 1 + x, \quad q(x) = 1, \quad f(x) = \sin x, \quad \alpha = 2, \quad \beta = -3, \quad \gamma = 3, u_b = 1.$$

Derive the linear system of the equations for the FEM approximation:

$$u_h = \sum \alpha_j \phi_j(x)$$

using the node points below

$$x_0 = 0, \quad x_2 = \frac{\pi}{4}, \quad x_3 = \pi/2, \quad x_1 = \frac{3\pi}{4}, \quad x_4 = \pi$$

and the linear basis functions. Find the finite element solution and plot it.

**Hint:** It is encouraged to use Matlab or Maple to generate the system of equations.