MA 587
Homework #1
Due Feb. 15

1. This problem deals with the following boundary value problem:

$$-u''(x) + u(x) = f(x), \qquad u(0) = u(1) = 0.$$

   (a) Show that the weak form (variational form) is given by

$$(u', v') + (u, v) = (f, v), \quad \forall\, v(x) \in H_0^1(0,1),$$

   where

$$\begin{aligned} (u, v) &= \int_0^1 u(x)v(x)dx, \\ H_0^1(0,1) &= \{v(x), \quad v(0) = v(1) = 0, \ \int_0^1 v^2 dx < \infty, \quad \int_0^1 v_x^2 dx < \infty\}. \end{aligned}$$

   (b) Assume given a uniform mesh $x_i = ih$, $i = 0, 1, \cdots, n$, $h = 1/n$, write down the linear system of equations using both the finite difference and finite element methods. Are they same?

   (c) Take $n = 3$, write down all the basis (hat) functions. Sketch or plot the basis functions, see also Problem 4 for a hint.

   (d) Derive the linear system of the equations for the FEM approximation:

$$u_h = \sum_{j=1}^{3} \alpha_j \phi_j(x)$$

   when $f(x) = 1$.

   (e) Solve the problem ($f(x) = 1$) and plot the finite element solution and the true solution in one plot. **hint:** he solution is $u(x) = C_1 \cosh x + C_2 \sinh x + 1$ or $u(x) = C_1 e^x + C_2 e^{-x} + 1$. Use the BC's to determine the constants C_1 and C_2.

   (f) Plot the error.

2. State the different formulations (D, V, and M) for solving $-(\beta(x)u'(x))' = f(x)$, $0 < x < 1$, and $u(0) = u(1) = 0$. Explain the conditions on $\beta(x)$, $f(x)$, and $u(x)$ that are necessary for each formulation. Explain the advantages and disadvantages when we use a finite difference or finite method for solving this problem.

Computer Projects: Download the Matlab files from

https://zhilin.math.ncsu.edu/TEACHING/MA587/index.html

Read the notes to understand what the codes are doing and test them.

3. This problem needs to modify *drive.m*, *f.m* and *soln.m*. Use the Matlab codes to solve

$$-u''(x) = f(x), \qquad u(0) = u(1) = 0.$$

Try two different triangulations: (a) the one given in *drive.m*; (b) the uniform grid $x_i = i*h$, $h = 1/M$, $i = 0, 1, \cdots, M$. Take $M = 10$. This can be done in Matlab using the command: $x = 0 : 0.1 : 1$.

Use the grids to solve the problems for the following $f(x)$ or exact $u(x)$ (derived analytically):

(a) $u(x) = \sin(\pi x)$, what is $f(x)$?

(b) $f(x) = x^3$, what is $u(x)$?

(c) (extra credit) $f(x) = \delta(x - 1/2)$, what is $u(x)$?

Make sure that the error is reasonably small.

4. This problem needs to modify *fem1d.m*, *drive.m*, *f.m* and *soln.m*. Assume we know that

$$\int_{x_i}^{x_{i+1}} \phi_i(x)\phi_{i+1}(x)dx = \frac{h}{6}$$

where $h = (x_{i+1} - x_i)$, ϕ_i and ϕ_{i+1} are the hat function centered at x_i and x_{i+1} respectively. Use the Matlab codes to solve

$$-u''(x) + u(x) = f(x), \qquad u(0) = u(1) = 0.$$

Try to use the uniform grid $x = 0 : 0.1 : 1$ in Matlab, for the following exact $u(x)$:

(a) $u(x) = \sin(\pi x)$, what is $f(x)$?

(b) $u(x) = x(1 - x)/2$, what is $f(x)$?