Stokes Equations

$$\begin{cases} \nabla p = \mu \Delta \vec{u} + \vec{F} & (x,y) \in \Omega \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}\big|_{\partial\Omega} = 0 \end{cases}$$

Renolds number is very small, inertia term is negnected, very viscous, small velocity. creeping flow.

<u>Finite element method:</u> A eliminate $p$.

$\vec{u} = (u_1, u_2)$

$$\iint_\Omega \nabla p \cdot \vec{v}\, dx\, dy = \iint_\Omega \mu \Delta \vec{u} \cdot \vec{v}\, dx\, dy + \iint \vec{F} \cdot \vec{v}\, dx\, dy$$

Require $\begin{cases} \nabla \cdot \vec{v} = 0 \\ \vec{v}\big|_{\partial\Omega} = 0 \end{cases}$, tough constraints.

$\nabla \vec{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial y} \end{bmatrix}$

$$\Rightarrow \int_{\partial\Omega} (\nabla p \cdot \vec{n})\, \vec{v}\, dx\, dy - \iint_\Omega p\, \nabla \cdot \vec{v}\, dx\, dy$$

$$= -\iint_\Omega \mu (\nabla \vec{u} \cdot \nabla \vec{v})\, dx\, dy + \int_{\partial\Omega} \mu \nabla \vec{u}\, \vec{v} + \iint_\Omega \vec{F} \cdot \vec{v}\, dx\, dy$$

$$\Rightarrow \mu \iint_\Omega \{ (\nabla u_1 \cdot \nabla v_1) + (\nabla u_2 \cdot \nabla v_2) \}\, dx\, dy = \iint \vec{F} \cdot \vec{v}\, dx\, dy$$

$$V = \{ \vec{v} \in [H_0^1(\Omega)]^2,\ \nabla \cdot \vec{v} = 0,\ \vec{v}\big|_{\partial\Omega} = 0 \}$$

Difficulty: How to find $V$ space,
Use the stream function $\varphi(x,y)$

$$\vec{v} = \operatorname{rot} \varphi = \left\langle \frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x} \right\rangle$$

then $\nabla\cdot\vec{v} = \operatorname{div}\vec{v} = \frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x} = 0$

Let $W_h$ be a finite dimensional subspace of $H_0^2(\Omega) = \{\vec{v}(x),\ \vec{v}(x)|_{\partial\Omega} = 0,\ \vec{v}(x) \in H^2(\Omega)\}$

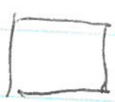
Then $V_h = \left\{\vec{v},\ \vec{v} = \begin{bmatrix} \frac{\partial\varphi}{\partial y} \\ -\frac{\partial\varphi}{\partial x} \end{bmatrix},\ \vec{v}|_{\partial\Omega} = 0,\ \varphi \in W_h \right\}$

Error estimate

$$\|\vec{u} - \vec{u}_h\|_{H^1(\Omega)} \le C h^4 \|\vec{u}\|_{H^5(\Omega)}$$

$W_h = \{\varphi(x,y),\ \frac{\partial\varphi}{\partial x}|_{\partial\Omega} = 0,\ \frac{\partial\varphi}{\partial y}|_{\partial\Omega} = 0,\ \varphi \in P_5(K),\ \varphi \in C^1(\Omega) \cap H^2(\Omega)\}$

= Note that: If the boundary conditions are periodic for all variables.



$$\vec{u}(a,y) = \vec{u}(b,y), \quad \vec{u}(x,c) = \vec{u}(x,d)$$
$$p(a,y) = p(b,y), \quad p(x,c) = p(x,d)$$

then we have

$$\Delta p = \nabla\cdot\mu\overbrace{\Delta\vec{u}}^{=0} + \nabla\cdot\vec{F}$$

$$\Rightarrow \begin{cases} \Delta p = \nabla\cdot\vec{F} \\ \mu\Delta u_1 = p_x + F_x \\ \mu\Delta u_2 = p_y + F_y \end{cases}$$

Method Mixed finite element method

Do not impose the incompressibility condition for $\vec{v}$.

$$\iint \nabla p \cdot \vec{v}\, dx\,dy = \iint \mu \overset{\nabla\cdot\nabla\vec{u}}{\Delta \vec{u}} \cdot \vec{v} + \iint \vec{F} \cdot \vec{v}\, dx\,dy$$

$$+ \int_{\partial\Omega} p\, \vec{n} \cdot \vec{v}\, ds - \iint_{\Omega} p\, \nabla \cdot \vec{v}\, dx\,dy$$

$$= \int_{\partial\Omega} \mu\, \nabla\vec{u} \cdot v\, ds - \iint_{\Omega} \nabla\vec{u} \cdot \nabla\vec{v}\, dx\,dy$$

$$+ \iint \vec{F} \cdot \vec{v}\, dx\,dy$$

$$\Rightarrow \iint_{\Omega} \nabla\vec{u} \cdot \nabla\vec{v}\, dx\,dy - \iint_{\Omega} p\, \nabla \cdot \vec{v}\, dx\,dy = \iint_{\Omega} \vec{F} \cdot \vec{v}\, dx\,dy$$

or $\quad (\nabla\vec{u}, \nabla\vec{v}) - (p, \operatorname{div}\vec{v}) = (\vec{F}, \vec{v})$

$$(\nabla w, \nabla v) = \sum_{i=1}^{2} \int_{\Omega} (\nabla w_i \cdot \nabla v_i)\, dx\,dy$$

second equation

$$\nabla \cdot \vec{u} = 0, \qquad \operatorname{div}\vec{u} = 0$$

$$\iint (q \cdot \operatorname{div}\vec{u})\, dx\,dy = 0$$

Impose additional equation for the pressure

$$\iint_{\Omega} p\, dx\,dy = 0$$

or $\quad p(x^*, y^*) = p_0 \qquad$ fixed at the point

$$V = \{ \vec{v} \in [H^1(\Omega)]^2, \ \vec{v}|_{\partial\Omega} = 0 \}$$

$$H = \{ q \in L_2(\Omega), \ \iint_{\Omega} q\, dx\,dy = 0 \}$$

Different sub-spaces, so it is called a mixed finite elements.

Stability requirement.

$$\|u_h\|_{H^1(\Omega)} + \|p_h\|_{L^2(\Omega)} \le C \underbrace{\sup_{v \in H^1(\Omega)} \frac{(f, v)_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}}}_{\|f\|_{H^{-1}(\Omega)}}$$

Then we have

$$\|u_h\|^2_{H^1(\Omega)} \le (f, u_h) \le \|f\|_{H^{-1}(\Omega)} \|u_h\|_{H^1(\Omega)}$$

or $\quad \|u_h\|_{H^1(\Omega)} \le \|f\|_{H^{-1}(\Omega)}$

Numerics: $V_h = \{ \vec{v} \in V : \vec{v}|_K \in [Q_2(K)]^2, \ \vec{v}(x)|_{\partial\Omega} = 0, \ \forall K \in T_h \}$.

$H_h = \{ \xi \in H : \xi|_K \in Q_0(K), \ \forall K \in T_h \}$.

or $\quad V_h = \{ \vec{v} \in V, \ \vec{v}|_K \in [Q_1(K)]^2, \ \forall K \in T_h \}$

$H_h = \{ \xi \in H, \ \xi|_K \in Q_0(K), \ \forall K \in T_h \}$.

or

saddle problem

$V_h \in \{ \vec{v} \in V; \ \vec{v}|_K \in [Q_2(K)]^2, \ \forall K \in T_h \}$

$H_h \in \{ \xi \in H, \ \xi|_K \in P_1(K), \ \forall K \in T_h \}$.

Reference: Numerical solution of PDE by the finite element method

Claes Johnson.

Cambridge University Press, 1994

Another method,

Least squares finite element method

$$\begin{cases} -u'' + 0.5u = 0 \\ u(0) = 0,\ u(\pi) = 0 \end{cases}$$

Don't exist

Eigenvalue Problem.

$$\begin{cases} -(pu')' + qu - \lambda u = 0 & a < x < b \\ u(a) = 0, & \alpha u(b) + \beta u'(b) = 0 \end{cases}$$

Weak form $\quad \int (p u'v' + (q - \lambda) uv)\, dx + \frac{\alpha}{\beta} p(b) u(b) v(b)$

$\quad\quad = \int f v\, dx + \cancel{\frac{\gamma}{\beta} p(b)} v(b)$

$a(u,u) \geq \alpha \|v\|_2^2$ is not true. The solution may not $\begin{cases} \text{exist} \\ \text{unique} \end{cases}$, Ex $\quad \begin{aligned} & -u'' + \lambda u = 0 \quad 0 \le x < \pi \\ & u(0) = 0 \quad u(x) = \sin \pi x \\ & u(\pi) = 0 \quad \text{or } u = 0 \end{aligned}$

$u = 0$ is a solution.
trivial solution

For certain $\lambda$, ~~there are~~ is another non-trivial solution. such $\lambda$ is called an eigenvalue of the Sturm-Liouville problem. the non-trivial solution is called the eigenvector.

$\lambda_1, \lambda_2, \cdots \lambda_n \ldots$

$u_1\ u_2 \cdots u_n \cdots \quad u_i \neq 0 \quad$ infinite dimension.

They form a basis for the ~~solution~~. Sturm-Liouville problem.

~~The solution~~ $\begin{cases} -(pu')' + qu = f \\ u(a) = u_a, \quad \alpha u(b) + \beta u'(b) = 0 \end{cases}$

can be expressed in terms of $u_i$

$$u(x) = \sum_{j=1}^{\infty} \alpha_j u_j(x)$$

$$\begin{aligned} & -u'' = f \\ & u(0) = 0,\ u(\pi) = 0 \end{aligned} \qquad u(x) = \sum_{j=1}^{\infty} \alpha_j \sin j\pi x$$

$$-u'' + \lambda u = 0$$

$$a(u,v) = \int (p u'v' + q u v)\,dx + \frac{\alpha}{\beta} p(b) u(b) v(b)$$

Numerics.

$$\begin{bmatrix} a(\varphi_1,\varphi_1) & \cdots & a(\varphi_1, \varphi_n) \\ & & \\ a(\varphi_n,\varphi_1) & \cdots & a(\varphi_n,\varphi_n) \end{bmatrix} U - \lambda \begin{bmatrix} (\varphi_1,\varphi_1) & (\varphi_1,\varphi_2) & & 0 \\ (\varphi_2,\varphi_1) & (\varphi_2,\varphi_2) & \ddots & \\ & & \ddots & \\ 0 & & & (\varphi_n,\varphi_n) \end{bmatrix} U = 0$$

$$A U - \lambda B U = 0$$

Generalized eigenvalue problem.

$$u(x) = \sum_{j=1}^{n} \alpha_j \varphi_j(x).$$

[V, D] = eig(A, B);

eigenvalues

eigenvectors.

Theorem: Let $u(x) \in C^2[a,b]$ be the exact solution of $\quad -u''(x) = f(x), \quad u(a) = u(b) = 0$.
$u_h(x)$ be the FEM solution using the hat basis function over a given triangulation. Then

(1) $\quad \|u-u_h\|_\infty = \max_{a \le x \le b} |u(x)-u_h(x)| \le h^2 \|u''\|_\infty / 8$

(2) $\quad u_h(x_j) = u(x_j), \quad j = 0, 1, \cdots, M.$

Proof: Define $\quad a(u,v) = \int_a^b u'v' \, dx$. Let $V_h$ be the finite dimensional space generated by the hat basis functions; $u_I$ is the interpolation function of $u(x)$. For any $v_h(x) \in V_h$, we have

$$\int_{x_i}^{x_{i+1}} (u-u_I)' v_h' \, dx = v_h'(u-u_I)\Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} (u-u_I) v_h'' \, dx = 0$$

since $u(x_j) = u_I(x_j)$, $j=0,1,\cdots,M$. $\quad v_h''(x) \equiv 0$ in $(x_i, x_{i+1})$.
Thus

$$a(u-u_I, v_h) = \int_a^b (u-u_I)' v_h' \, dx = \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} (u-u_I)' v_h' \, dx = 0$$

We also know that

$$a(u-u_h, v_h) = 0, \quad \text{for any } v_h \in V_h$$

Subtract to get

$$a(u-u_I-u+u_h, v_h) = 0$$

That is $\quad a(u_h - u_I, v_h) = 0 \quad$ for any $v_h \in V_h$

Take $v_h = u_h - u_I$, we have

$$a(u_h - u_I, u_h - u_I) = 0$$