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Appropriate Gaussian quadrature formulae for triangles

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ABSTRACT

This paper mainly presents higher order Gaussian quadrature formulae for numerical integration over the triangular surfaces. In order to show the exactness and efficiency of such derived quadrature formulae, it also shows first the effective use of available Gaussian quadrature for square domain integrals to evaluate the triangular domain integrals. Finally, it presents $n \times n$ points and $\frac{n(n+1)}{2} - 1$ points (for $n > 1$) Gaussian quadrature formulae for triangle utilizing n-point one-dimensional Gaussian quadrature. By use of simple but straightforward algorithms, Gaussian points and corresponding weights are calculated and presented for clarity and reference. The proposed $\frac{n(n+1)}{2} - 1$ points formulae completely avoids the crowding of Gaussian points and overcomes all the drawbacks in view of accuracy and efficiency for the numerical evaluation of the triangular domain integrals of any arbitrary functions encountered in the realm of science and engineering.

Keywords: Extended Gaussian Quadrature, Triangular domain, Numerical accuracy, Convergence

1. Introduction

The integration theory extends from real line to the plane and three-dimensional spaces by the introduction of multiple integrals. Integration procedures on finite domains underlie physically acceptable averaging process in engineering. In probabilistic estimations and in spatially discretized approximations, e.g., finite and boundary-element methods, evaluation of integrals over arbitrary-shaped domain Ω are the pivotal task. In practice, most of the integrals (encountered frequently) either cannot be evaluated analytically or the evaluations are very lengthy and tedious. Thus, for simplicity numerical integration methods are preferred and the methods extensively employ the Gaussian quadrature technique that was originally designed for one dimensional cases and the procedure naturally extends to two and three-dimensional rectangular domains according to the notion of the Cartesian product. Gaussian quadratures are considered as the best method of integrating polynomials because they guarantee that they are exact for polynomials less than a specified degree.

In order to obtain the result with the desired accuracy, Gaussian integration points and weights necessarily increase and there is no computational difficulty except time in evaluating any domain integral when the two and three-dimensional regions are bounded respectively, by systems of parallel lines and parallel planes.

Analysts cannot ignore at all the randomness in material properties and uncertainty in geometry that are frequently encountered in complex engineering systems. Specifically, the vital components are rated during quality control inspections according to reliability indices calculated from the average probability density functions that model failure. This entails the evaluation of an integral of the function (say joint probability frequency function) over the volume Ω of the component. In general, the Ω-shape-class is very irregular in two and three-dimensional geometry. For non-parallelogram quadrilateral, very frequent in finite-element modelling, there is no consistent procedure to select the sampling point to implement a Gaussian quadrature on the entire element.

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Special integration schemes, e.g., reduced integration over quadrilaterals have been successfully developed in [1] and are widely used in commercial programs. There is no methodical way to design such approximate integration schemes for polygons with more than four sides. An attempt to distribute the sampling points according to the governing perspective transformation fails to assure the error order germane to the quadrature formula. The reason can be traced to the crowding of quadrature points and this numerical computational difficulty persists in all non-parallelogram polygonal finite elements [2]. A considerable amount of research has been performed to attain perfect results of domain integration for plane quadrilateral elements where numerical quadrature techniques are employed [3]. The accuracy of a selected quadrature strategy is indicated by compliance with the patch test proposed in [4].

The overall error in a finite element calculation can be reduced by not relying so heavily on artificial tessellation, which requires the deployment of elements with large number of sides. An elegant systematic procedure to yield shape functions for convex polygons of arbitrary number of sides developed in [5] by which the energy density can be obtained in closed algebraic form in terms of rational polynomials. However, a direct Gaussian quadrature scheme to numerically evaluate the domain integral on n-sided polygons cannot be constructed to yield the exact results, even on convex quadrilaterals. In two-dimension, n-sided polygons can be suitably discretized with linear triangles rather than quadrilaterals (Fig. 1(a-b)) and hence triangular elements are widely used in finite element analysis. Another advantage is to be mentioned that there is no difficulty with triangular elements as the exact shape functions are available and the quadrature formulas are also exact for the polynomial integrands [6].

Integration schemes based on weighted residuals are prone to instability since the accuracy goal cannot be controlled. In deterministic cases the underlying averaging process may be inconsistent, which was stated as a variational crime [7]. In stochastic differential equation literature [8, 9], such averaging processes are termed dishonest [10]. Thus, the high accuracy integration method is demanded and it is meaningful when the shape functions are the very best. Therefore, there has been considerable interest in the area of numerical integration schemes over triangles [11] to [24]. It is explicitly shown in [21, 24] that the most accurate rules are not sufficient to evaluate the triangular domain integrals and for some element geometry these rules are not reliable also.

To address all these short comings, to make a proper balance between accuracy and efficiency and to avoid the crowding of quadrature points we have proposed $n \times n$ points and $\frac{n(n+1)}{2} - 1$ points higher order Gaussian quadrature formulae to evaluate the triangular domain integrals. It is thoroughly investigated that the $\frac{n(n+1)}{2} - 1$ point formulae are appropriate in view of accuracy and efficiency and hence we believe that the formulae will find better place in numerical solution procedure of continuum mechanics problems.

2. Problem Statement

In finite and boundary element methods for two-dimensional problems, a pivotal task is to evaluate the integral of a function f:

$$I_1 = \iint_{\Omega} f \, d\Omega; \quad \Omega\text{: element domain} \tag{2.1}$$

Observe that I_1 can be calculated as a sum of integrals evaluated over simplex divisions Δ_i :

$$\Omega = \bigcup_i \Delta_i; \quad \Delta_i: \text{ completely covers } \Omega \tag{2.2}$$

Δ_i= triangle for two-dimensional domain (see Fig. 1(a-b)). Now equation (2.1) can be written as

$$I_1 = \iint_{\Omega} f \, d\Omega = \sum_i \iint_{\Delta_i} f \, d\Delta_i \tag{2.3}$$

To evaluate the integral I_1 in equation (2.3), it is now required to evaluate the triangular domain integral

$$I_2 = \iint\limits_{\Delta} f(x, y)\, dx\, dy; \quad \Delta: \text{ triangle (arbitrary)} \tag{2.4}$$

Integration over triangular domains is usually carried out in normalized co-ordinates. To perform the integration, first map one vertex (vertex 1) to the origin, the second vertex (vertex 2) to point (1, 0) and the third vertex (vertex 3) to point (0, 1), (see Fig 2(a), (b)). This transformation is most easily accomplished by use of shape functions as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \tag{2.5}$$

where

$$N_1(s, t) = 1 - s - t, \quad N_2(s, t) = s, \quad N_3(s, t) = t \tag{2.6}$$

The original and the transformed triangles are shown in Fig. 2. Form Eq. (5) using Eq. (6), we obtain

$$\begin{aligned} x(s, t) &= x_1 + (x_2 - x_1)s + (x_3 - x_1)t \\ y(s, t) &= y_1 + (y_2 - y_1)s + (y_3 - y_1)t \end{aligned} \tag{2.7}$$

and hence

$$\frac{\partial(x, y)}{\partial(s, t)} = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) = Area \tag{2.8}$$

Finally, equation (2.4) reduces to

$$I_2 = Area \int_{s=0}^{1} \int_{t=0}^{1-s} f(x(s, t), y(s, t))\, dt\, ds \tag{2.9}$$

One can simply verify that

$$I_2 = Area \int_{t=0}^{1} \int_{s=0}^{1-t} f(x(s, t), y(s, t))\, ds\, dt \tag{2.10}$$

Here, we wish to mention that the evaluation of integrals I_2 in equation (2.9) and in equation (2.10) by the existing Gaussian quadrature (i.e. 7-point and 13-point) will yield the same results. Thus, any one of these two can be evaluated numerically. Influences of these integrals will be investigated later to present new quadrature formulae for triangles.

3. Numerical evaluation procedures

In this section, we wish to describe three procedures to evaluate the integral I_2 numerically and new Gaussian quadrature formulae for triangles.

3.1 Procedure-1

Use of Gaussian quadrature for triangle (GQT): Gaussian quadrature for triangle in [11] to [24] can be employed as

$$I_2 = Area \sum_{i=1}^{NGP} \sum_{j=1}^{NGP} W_i W_j f(x(s_i, t_j), y(s_i, t_j)) \tag{3.1}$$

where (s_i, t_j) are the ij-th sampling points W_i, W_j are corresponding weights and NGP denotes the number of gauss points in the formula. It is thoroughly investigated that in some cases available Gaussian quadrature for triangle cannot evaluate the integral I_2 exactly [11, 21, 24].

3.2 Procedure-2

Use of Gaussian quadrature for square (IOST): Integration over the normalized (unit) triangle can be calculated as a sum of integrals evaluated over three quadrilaterals (fig-3a,b).

$$\begin{aligned} I_2 &= \int_{s=0}^{1}\int_{t=0}^{1-s} f(x(s,t),y(s,t))\frac{\partial(x,y)}{\partial(s,t)}\,dt\,ds \\ &= \sum_{i=1}^{3}\iint_{e_i} f(x(s,t),y(s,t))\,\frac{\partial(x,y)}{\partial(s,t)}\,dt\,ds \\ &= \frac{\text{Area}}{96}\int_{-1}^{1}\int_{-1}^{1}\left[f(X_1,Y_1)(4-\xi+\eta)+f(X_2,Y_2)(4-\xi-\eta)+f(X_3,Y_3)(4+\xi-\eta)\right]d\xi\,d\eta \end{aligned} \tag{3.2}$$

Equation (3.2) is obtained after transforming each quadrilaterals in to a square in (ξ,η) space where

$$\begin{aligned} X_1 &= \frac{1}{24}[a_{11}+a_{12}\xi+a_{13}\eta+a_{14}\xi\,\eta] \quad & Y_1 &= \frac{1}{24}[b_{11}+b_{12}\xi+b_{13}\eta+b_{14}\xi\eta] \\ X_2 &= \frac{1}{24}[a_{21}+a_{22}\xi+a_{23}\eta+a_{24}\xi\eta], \quad & Y_2 &= \frac{1}{24}[b_{21}+b_{22}\xi+b_{23}\eta+b_{24}\xi\eta] \\ X_3 &= \frac{1}{24}[a_{31}+a_{32}\xi+a_{33}\eta+a_{34}\xi\eta], \quad & Y_3 &= \frac{1}{24}[b_{31}+b_{32}\xi+b_{33}\eta+b_{34}\xi\eta] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} a_{11} &= 5x_1+5x_2+14x_3 & b_{11} &= 5y_1+5y_2+14y_3 \\ a_{12} &= -x_1+5x_2-4x_3 & b_{12} &= -y_1+5y_2-4y_3 \\ a_{13} &= -5x_1+x_2+4x_3 & b_{13} &= -5y_1+y_2+4y_3 \\ a_{14} &= x_1+x_2-2x_3 & b_{14} &= y_1+y_2-2y_3 \\ a_{21} &= 14x_1+5x_2+5x_3 & b_{21} &= 14y_1+5y_2+5y_3 \\ a_{22} &= -4x_1+5x_2-x_3 & b_{22} &= -4y_1+5y_2-y_3 \\ a_{23} &= -4x_1-x_2+5x_3 & b_{23} &= -4y_1-y_2+5y_3 \\ a_{24} &= 2x_1-x_2-x_3 & b_{24} &= 2y_1-y_2-y_3 \\ a_{31} &= 5x_1+14x_2+5x_3 & b_{31} &= 5y_1+14y_2+5y_3 \\ a_{32} &= -5x_1+4x_2+x_3 & b_{32} &= -5y_1+4y_2+y_3 \\ a_{33} &= -x_1-4x_2+5x_3 & b_{33} &= -y_1-4y_2+5y_3 \\ a_{34} &= x_1-2x_2+x_3 & b_{34} &= y_1-2y_2+y_3 \end{aligned}$$

Now right hand side of equation (3.2) with equations (3.3) can be evaluated by use of available higher order Gaussian quadrature for square. For clarity, we mention that each quadrilaterals in Fig. 3(b) is transformed into 2-square in $(\xi,\eta) \in \{(-1,-1),(1,-1),(1,1),(-1,1)\}$ space through isoperimetric transformation to get the integral I_2 in equation (3.2).

3.3 Procedure-3:

In this section, we wish to present two new techniques to evaluate the integrals over the triangular surface and to calculate Gaussian points and corresponding weights for triangle.

Using mathematical transformation equations:

$$s = \frac{1+\xi}{2}, \quad t = \left(1-\frac{1+\xi}{2}\right)\left(\frac{1+\eta}{2}\right) = \frac{1}{4}(1-\xi)(1+\eta) \tag{3.4}$$

the integral I_2 of equation (2.9) is transformed into an integral over the surface of the standard square $\{(\xi, \eta)| - 1 \leq \xi, \eta \leq 1\}$ and the equation (2.7) reduces to

$$x = x_1 + \frac{1}{2}(x_2 - x_1)(1 + \xi) + \frac{1}{4}(x_3 - x_1)(1 - \xi)(1 + \eta)$$
$$y = y_1 + \frac{1}{2}(y_2 - y_1)(1 + \xi) + \frac{1}{4}(y_3 - y_1)(1 - \xi)(1 + \eta) \tag{3.5}$$

Now the determinant of the Jacobean and the differential area are:

$$\frac{\partial(s,t)}{\partial(\xi,\eta} = \frac{\partial s}{\partial \zeta}\frac{\partial t}{\partial \eta} - \frac{\partial s}{\partial \eta}\frac{\partial t}{\partial \zeta} = \frac{1}{8}(1 - \zeta) \tag{3.6}$$

$$ds\ dt = dt\ ds = \frac{\partial(s,t)}{\partial(\xi,\eta}\ d\xi\ d\eta = \frac{1}{8}\ (1 - \xi)\ d\xi\ d\eta \tag{3.7}$$

Now using equation (3.4) and equation (3.7) into equation (2.9), we get

$$I_2 = Area \int_{-1}^{1}\int_{-1}^{1} f\left(x\left(\frac{1+\xi}{2}, \frac{(1-\xi)(1+\eta)}{4}\right), y\left(\frac{1+\xi}{2}, \frac{(1-\xi)(1+\eta)}{4}\right)\right)\frac{1-\xi}{8}\ d\xi d\eta$$
$$= Area \int_{-1}^{1}\int_{-1}^{1} f\left(\frac{1+\xi}{2}, \frac{(1-\xi)(1+\eta)}{4}\right)\frac{1-\xi}{8}\ d\xi d\eta \tag{3.8}$$

In order to evaluate the integral I_2 in equation (3.8) efficient Gaussian quadrature co-efficient (points and weights) are readily available so that any desired accuracy can be readily obtained [21, 24].

3.3.1 New quadrature formula

GQUTS:

In this section we are straightly computing Gaussian quadrature formula for unit triangles (GQUTS). The Gauss points are calculated simply for i = 1, m and j = 1, n. Thus the $m \times n$ points Gaussian quadrature formula for (3.8) gives

$$I_2 = Area \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{1-\xi_i}{8}\right)W_iW_jf\left(\frac{1+\xi_i}{2}, \frac{(1-\xi_i)(1+\eta_j)}{4}\right)$$
$$= Area \sum_{r=1}^{m\times n} G_r f(u_r, v_r) \tag{3.9}$$

where (u_r, v_r) are the new Gaussian points, G_r is the corresponding weights for triangles. Again, if we consider the integral I_2 of equation (2.10) and substitute

$$t = \frac{1+\eta}{2}, \quad s = \left(1 - \frac{1+\eta}{2}\right)\left(\frac{1+\xi}{2}\right)$$

Then one can obtain (on the same line of equation (3.9)))

$$I_2 = Area \int_{-1}^{1}\int_{-1}^{1} f\left(\frac{(1+\xi)(1-\eta)}{4}, \frac{1+\eta}{2}\right)\frac{1-\eta}{8}d\xi\ d\eta$$
$$= Area \sum_{r=1}^{m\times n} G'_r f(u'_r, v'_r) \tag{3.10}$$

where G'_r and (u'_r, v'_r) are respectively weights and Gaussian points for triangle.

All the Gaussian points and corresponding weights can be calculated simply using the following algorithm:

step 1. $r \to 1$

step 2. $i = 1, m$

step 3. $j = 1, n$

$$G_r = \frac{(1-\zeta_i)}{8} W_i W_j, \quad u_r = \frac{1+\zeta_i}{2}, \quad v_r = \frac{(1-\zeta_i)(1+\eta_j)}{4}$$

$$G_r^{/} = \frac{(1-\eta_j)}{8} W_i W_j, \quad u_r^{/} = \frac{(1+\zeta_i)(1-\eta_j)}{4}, \quad v_r^{/} = \frac{1+\eta_j}{2}$$

step 4. compute step 3

step 5. compute step 2

For clarity and reference, computed Gauss points and weights (for $n = 2, 3, 7$) based on above algorithm listed in table-1 and Fig. 4a shows the distribution of Gaussian points for $n = 10$. In figure-4a, it is seen that there is a crowding of gauss points at least at one point within the triangle and that is one of the major causes of error germen in the calculation. To avoid this crowding further modification is needed. This modification is obtained in the next section.

Table 1: Computed weights G and corresponding Gauss points (u, v) for $n \times n$ point method (GQUTS).

n	G	u	v
2	0.5283121635D-01	0.1666666667D+00	0.7886751346D+00
	0.1971687836D+00	0.6220084679D+00	0.2113248654D+00
	0.5283121635D-01	0.4465819874D-01	0.7886751346D+00
	0.1971687836D+00	0.1666666667D+00	0.2113248654D+00
3	0.9876542474D-01	0.2500000000D+00	0.5000000000D+00
	0.1391378575D-01	0.5635083269D-01	0.8872983346D+00
	0.1095430035D+00	0.4436491673D+00	0.1127016654D+00
	0.6172839460D-01	0.4436491673D+00	0.5000000000D+00
	0.8696116674D-02	0.1000000000D+00	0.8872983346D+00
	0.6846438175D-01	0.7872983346D+00	0.1127016654D+00
	0.6172839460D-01	0.5635083269D-01	0.5000000000D+00
	0.8696116674D-02	0.1270166538D-01	0.8872983346D+00
	0.6846438175D-01	0.1000000000D+00	0.1127016654D+00
7	0.2183621219D-01	0.2500000000D+00	0.5000000000D+00
	0.1185259869D-01	0.1485387122D+00	0.7029225757D+00
	0.2804474024D-01	0.3514612878D+00	0.2970774243D+00
	0.3777048400D-02	0.6461720360D-01	0.8707655928D+00
	0.2544928909D-01	0.4353827964D+00	0.1292344072D+00
	0.3442812316D-03	0.1272302191D-01	0.9745539562D+00
	0.1318557174D-01	0.4872769781D+00	0.2544604383D-01
	0.1994866947D-01	0.3514612878D+00	0.5000000000D+00
	0.1082804890D-01	0.2088224283D+00	0.7029225757D+00
	0.2562052651D-01	0.4941001474D+00	0.2970774243D+00
	0.3450556783D-02	0.9084178238D-01	0.8707655928D+00
	0.2324942860D-01	0.6120807933D+00	0.1292344072D+00
	0.3145212381D-03	0.1788659867D-01	0.9745539562D+00
	0.1204579851D-01	0.6850359770D+00	0.2544604383D-01
	0.1994866947D-01	0.1485387122D+00	0.5000000000D+00
	0.1082804890D-01	0.8825499604D-01	0.7029225757D+00
	0.2562052651D-01	0.2088224283D+00	0.2970774243D+00
	0.3450556783D-02	0.3839262482D-01	0.8707655928D+00
	0.2324942860D-01	0.2586847995D+00	0.1292344072D+00
	0.3145212381D-03	0.7559445160D-02	0.9745539562D+00
	0.1204579851D-01	0.2895179792D+00	0.2544604383D-01
	0.1461316874D-01	0.4353827964D+00	0.5000000000D+00
	0.7931962886D-02	0.2586847995D+00	0.7029225757D+00
	0.1876802249D-01	0.6120807933D+00	0.2970774243D+00
	0.2527665748D-02	0.1125328752D+00	0.8707655928D+00
	0.1703110194D-01	0.7582327176D+00	0.1292344072D+00
	0.2303989213D-03	0.2215753944D-01	0.9745539562D+00
	0.8824011376D-02	0.8486080534D+00	0.2544604383D-01
	0.1461316874D-01	0.6461720360D-01	0.5000000000D+00
	0.7931962886D-02	0.3839262482D-01	0.7029225757D+00
	0.1876802249D-01	0.9084178238D-01	0.2970774243D+00
	0.2527665748D-02	0.1670153200D-01	0.8707655928D+00
	0.1703110194D-01	0.1125328752D+00	0.1292344072D+00
	0.2303989213D-03	0.3288504390D-02	0.9745539562D+00
	0.8824011376D-02	0.1259459028D+00	0.2544604383D-01
	0.6764926484D-02	0.4872769781D+00	0.5000000000D+00
	0.3671971955D-02	0.2895179792D+00	0.7029225757D+00
	0.8688347794D-02	0.6850359770D+00	0.2970774243D+00
	0.1170141347D-02	0.1259459028D+00	0.8707655928D+00
	0.7884268950D-02	0.8486080534D+00	0.1292344072D+00
	0.1066593969D-03	0.2479854268D-01	0.9745539562D+00
	0.4084931154D-02	0.9497554135D+00	0.2544604383D-01
	0.6764926484D-02	0.1272302191D-01	0.5000000000D+00
	0.3671971955D-02	0.7559445160D-02	0.7029225757D+00
	0.8688347794D-02	0.1788659867D-01	0.2970774243D+00
	0.1170141347D-02	0.3288504390D-02	0.8707655928D+00
	0.7884268950D-02	0.2215753944D-01	0.1292344072D+00
	0.1066593969D-03	0.6475011465D-03	0.9745539562D+00
	0.4084931154D-02	0.2479854268D-01	0.2544604383D-01

3.3.2 New quadrature formula

GQUTM:

It is clearly noticed in the equation (3.9) that for each i ($i = 1, 2, 3, ..., m$), j varies from 1 to n and hence at the terminal value $i = m$ there are n crowding points as shown in Table-1 and fig-4a. To overcome this situation, we can use the advantage of equation (3.9) by making j dependent on i for the calculation of new gauss points and corresponding weights. To do so, we wish to calculate gauss points and weights for $i = 1, m-1$ and $j = 1, m+1-i$ that is $\frac{m(m+1)}{2} - 1$ points Gaussian quadrature formulae from equation (3.9) as:

$$\begin{aligned} I_2 &= Area \sum_{i=1}^{m-1} \sum_{j=1}^{m+1-i} (\frac{1-\xi_i}{8}) W_i W_j f\{\frac{1+\xi_i}{2}, \frac{(1-\xi_i)(1+\eta_j)}{4}\} \\ &= Area \left\{ \sum_{r=1}^{\frac{m(m+1)}{2}-1} L_r f(p_r, q_r) \right\} \end{aligned} \tag{3.11}$$

where (p_r, q_r) are the new Gaussian points, L_r is the corresponding weights for triangles. Similarly, we can write equation (3.10) as:

$$\begin{aligned} I_2 &= Area \int_{-1}^{1} \int_{-1}^{1} f\left[\frac{(1+\xi)(1-\eta)}{4}, \frac{1+\eta}{2}\right] \frac{1-\eta}{8}\, d\xi\, d\eta \\ &= Area \left\{ \sum_{r=1}^{\frac{m(m+1)}{2}-1} L'_r f(p'_r, q'_r) \right\} \end{aligned} \tag{3.12}$$

where L'_r and (p'_r, q'_r) are respectively weights and Gaussian points for triangle. All the Gaussian points and corresponding weights can be calculated simply using the following algorithm:

step 1. $r \to 1$

step 2. $i = 1, m-1$

step 3. $j = 1, m+1-i$

$L_r = \frac{(1-\zeta_i)}{8} W_i W_j, \quad p_r = \frac{1+\zeta_i}{2}, \quad q_r = \frac{(1-\zeta_i)(1+\eta_j)}{4}$

step 4. $j = 1, m-1$

step 5. $i = 1, m+1-j$

$L'_r = \frac{(1-\eta_i)}{8} W_i W_j, \quad p'_r = \frac{(1+\zeta_i)(1-\eta_j)}{4}, \quad q'_r = \frac{1+\eta_j}{2}$

$r = r+1$

step 6. compute step 3, step 2

step 7. compute step 5, step 4

Thus, the new $\frac{m(m+1)}{2} - 1$ points Gaussian quadrature formulae is now obtained which is crowding free. For clarity and reference, computed Gauss points and weights (for $m = 5, 9$) based on above algorithm listed in Table-2 and Fig. 4b shows the distribution of Gaussian points for $m = 10$ i.e. 54-points formula.

4. Application Examples

To show the accuracy and efficiency of the derived formulae, following examples with known results are considered:

Table 2: Computed Gauss points (p, q) and corresponding weights L for $\frac{n(n+1)}{2} - 1$ point method GQUTM.

n	p	q	L
n=5	6.943184420297371E-002	4.365302387072518E-002	1.917346464706755E-002
	6.943184420297371E-002	0.214742881469342	3.873334126144628E-002
	6.943184420297371E-002	0.465284077898513	4.603770904527855E-002
	6.943184420297371E-002	0.715825274327684	3.873334126144628E-002
	6.943184420297371E-002	0.886915131926301	1.917346464706755E-002
	0.330009478207572	4.651867752656094E-002	3.799714764789616E-002
	0.330009478207572	0.221103222500738	7.123562049953998E-002
	0.330009478207572	0.448887299291690	7.123562049953998E-002
	0.330009478207572	0.623471844265867	3.799714764789616E-002
	0.669990521792428	3.719261778493340E-002	2.989084475992800E-002
	0.669990521792428	0.165004739103786	4.782535161588505E-002
	0.669990521792428	0.292816860422638	2.989084475992800E-002
	0.930568155797026	1.467267513102734E-002	6.038050853208200E-003
	0.930568155797026	5.475916907194637E-002	6.038050853208200E-003
n=9	1.985507175123191E-002	1.560378988162790E-002	2.015983497663207E-003
	1.985507175123191E-002	8.035663927218221E-002	4.480916044841641E-003
	1.985507175123191E-002	0.189476014677302	6.464359484621604E-003
	1.985507175123191E-002	0.331164789916112	7.747662769908149E-003
	1.985507175123191E-002	0.490072464124384	8.191474625434276E-003
	1.985507175123191E-002	0.648980138332656	7.747662769908149E-003
	1.985507175123191E-002	0.790668913571466	6.464359484621604E-003
	1.985507175123191E-002	0.899788288976586	4.480916044841641E-003
	1.985507175123191E-002	0.964541138367140	2.015983497663207E-003
	0.101666761293187	1.783647091104033E-002	5.055663745070170E-003
	0.101666761293187	9.133063094134081E-002	1.110639128725685E-002
	0.101666761293187	0.213115003430640	1.566747257514398E-002
	0.101666761293187	0.366773901111335	1.811354111938598E-002
	0.101666761293187	0.531559337595478	1.811354111938598E-002
	0.101666761293187	0.685218235276173	1.566747257514398E-002
	0.101666761293187	0.807002607765473	1.110639128725685E-002
	0.101666761293187	0.880496767795773	5.055663745070170E-003
	0.237233795041836	1.940938228235618E-002	7.745946956361961E-003
	0.237233795041836	9.857563833019303E-002	1.673231410555364E-002
	0.237233795041836	0.226600619520678	2.284153446586376E-002
	0.237233795041836	0.381383102479082	2.500282281756943E-002
	0.237233795041836	0.536165585437487	2.284153446586376E-002
	0.237233795041836	0.664190566627971	1.673231410555364E-002
	0.237233795041836	0.743356822675808	7.745946956361961E-003
	0.408282678752175	1.997947907913758E-002	9.191827856850984E-003
	0.408282678752175	0.100234137152044	1.935542449754594E-002
	0.408282678752175	0.225261107830170	2.510431683577024E-002
	0.408282678752175	0.366456213417655	2.510431683577024E-002
	0.408282678752175	0.491483184095780	1.935542449754594E-002
	0.408282678752175	0.571737842168687	9.191827856850984E-003
	0.591717321247825	1.915257191055202E-002	8.770885597453929E-003
	0.591717321247825	9.421749319819557E-002	1.771853503082167E-002
	0.591717321247825	0.204141339376088	2.105991205229386E-002
	0.591717321247825	0.314065185553979	1.771853503082167E-002
	0.591717321247825	0.389130106841623	8.770885597453929E-003
	0.762766204958164	1.647157989702492E-002	6.471997505236908E-003
	0.762766204958164	7.828940091495819E-002	1.213345702759751E-002
	0.762766204958164	0.158944394126877	1.213345702759751E-002
	0.762766204958164	0.220762215144811	6.471997505236908E-003
	0.898333238706813	1.145801331145764E-002	3.140105492486528E-003
	0.898333238706813	5.083338064659329E-002	5.024168787978471E-003
	0.898333238706813	9.020874798172894E-002	3.140105492486528E-003
	0.980144928248768	4.195870365439417E-003	5.024749628293684E-004
	0.980144928248768	1.565920138579250E-002	5.024749628293684E-004

$$I_1 = \int_{y=0}^{1} \int_{x=0}^{1-y} (x+y)^{\frac{1}{2}} dx\, dy = 0.4$$
$$I_2 = \int_{y=0}^{1} \int_{x=0}^{1-y} (x+y)^{-\frac{1}{2}} dx\, dy = 0.6666667$$

$$I_3 = \int_{y=0}^{1} \int_{x=0}^{y} (x^2+y^2)^{-\frac{1}{2}} dx\, dy = 0.881373587$$
$$I_4 = \int_{y=0}^{1} \int_{x=0}^{y} exp^{|x+y-1|} dx\, dy = 0.71828183$$

Computed values (by use of three procedures) are summarized in Table-3.
Some important remarks from the Table-3 are:

- Usual Gauss quadrature (GQT) for triangles e.g. 7-point and 13-point rules cannot evaluate the integral of non-polynomial functions accurately.

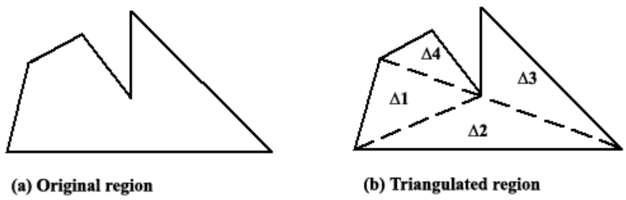

Figure 1: Triangulation of the domain of integral

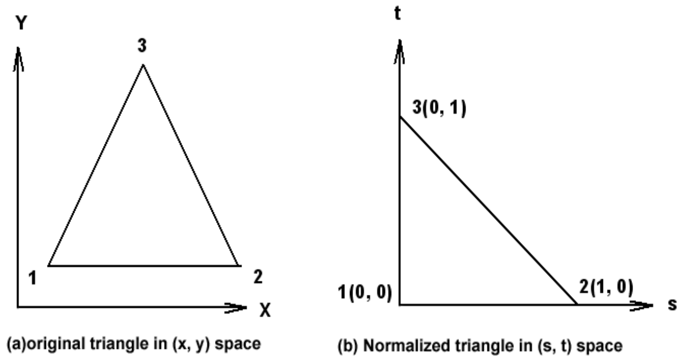

Figure 2: The original and transformed triangle

• Splitting unit triangle into quadrilaterals (IOST) provides the way of using Gaussian quadrature for square and the convergence rate is slow but satisfactory in view of accuracy.

• New Gaussian quadrature formulae for triangle (GQUTS and GQUTM) are exact in view of accuracy and efficiency and (GQUTM) is faster.

Again, we consider the following integrals of rational functions due to [24] to test the influences of formulae in equations (3.9), (3.10), (3.11) and (3.12) as described in procedure-3. Consider

$$I^{p,q} = \int_{y=0}^{1} \int_{x=0}^{1-y} \frac{x^p y^q}{\alpha + \beta x + \gamma y} \, dx \, dy$$

Example-1: $I^{r,0} = \int_{y=0}^{1} \int_{x=0}^{1-y} \frac{x^r}{0.375 - 0.375\, x} \, dx \, dy$

Example-2: $I^{0,r} = \int_{y=0}^{1} \int_{x=0}^{1-y} \frac{y^r}{0.375 - 0.375\, y} \, dx \, dy$

Example-3: $I^{0,0} = \int_{y=0}^{1} \int_{x=0}^{1-y} \frac{1}{12 + 21.53679831x - 8.0821067231y} \, dx \, dy$

Example-4: $I^{0,0} = \int_{y=0}^{1} \int_{x=0}^{1-y} \frac{1}{12 + 9.941125498(x+y)} \, dx \, dy$

Results are summarized in Tables-(4, 5, 6, 7).

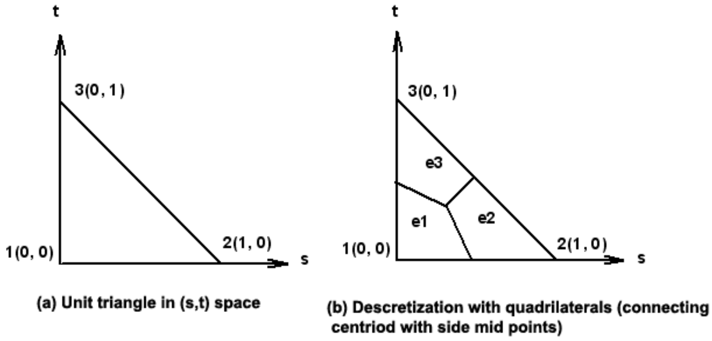

Figure 3: Unit triangle splited into three quadrilaterals

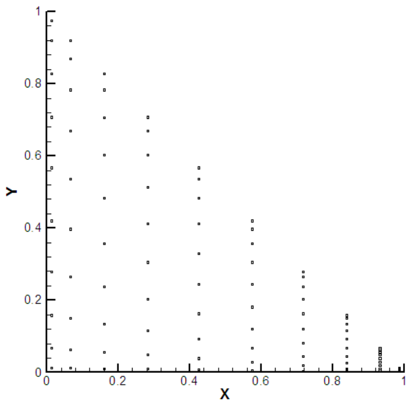

Figure 4

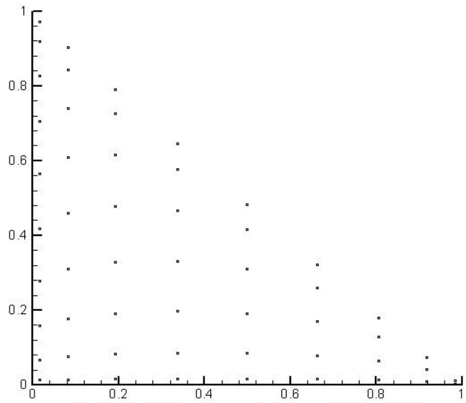

Figure 5

Some important comments may be drawn from the tables (4 - 7). In tables (4 - 7) for method GQUTS, Formula 1 is for equation (3.9) and Formula 2 is for equation (3.10), for method GQUTM, Formula 1 is for equation (3.11) and Formula 2 is for equation (3.12). These tables substantiated the influences of numerical evaluation of the integrals as described in section-3.

- For the integrand $\frac{x^r}{\alpha+\beta x+\gamma y}$ with $\beta \neq \gamma = 0$ first formulae in equation (3.9) and (3.11) described in procedure-3 is more accurate and rate of convergence is higher. But the new formula in equation (3.11) requires very less computational effort.

- Similarly for the integrand $\frac{y^r}{\alpha+\beta x+\gamma y}$ with $\gamma \neq \beta = 0$ second formula in equation (3.10) and (3.12) as described in procedure-3 is more accurate and convergence is higher. Here also the new formula in equation (3.12) requires very less computational effort.

- Similar influences of these formulae in procedure-3 may be observed for different conditions on β, γ.

- General Gaussian quadrature e.g. 7-point and 13-point rules cannot evaluate the integral of rational functions accurately.

It is evident that the new formulae e.g. equation (3.11) and (3.12) are very fast and accurate in view of accuracy and equally applicable for any geometry that is for different values of α, β and γ. We recommend this is appropriate quadrature scheme for triangular domain integrals encountered in science and engineering.

Also the method is tested on the integral of all monomials $x^i y^j$ where i , j are non-negative integers such that $i + j \leq 30$. In table 8, we present the absolute error over corresponding monomials integrals

Table 3: Calculated values of the integrals I_1, I_2, I_3, I_4

Method	Points	Test example			
		I_1	I_2	I_3	I_4
GQT	7×7	0.4001498818	0.6606860757	0.8315681219	0.6938790083
	13×13	0.4000451564	0.66370582580	0.85017383098	0.72387170791
IOST	7×7	0.4000006725	0.6664256210	0.8755247309	0.7178753433
	10×10	0.4000001234	0.6665789279	0.8783900003	0.7180745324
GQUTS	7×7	0.4000037499	0.6659893974	0.8696444431	0.7184323939
	10×10	0.4000006929	0.6664193645	0.8753981854	0.7182531970
GQUTM	54	0.4000009417	0.6663718426	0.8742865042	0.7175459725
	90	0.4000002469	0.6665339400	0.8772635782	0.7180958214
Exact Value		0.4	0.6666667	0.881373587	0.71828183

Table 4: Computed results of Example -1 for r=2, r=4, r=6.

Method	Points	Computed value of $I^{r,0}$					
		r=2		r=4		r=6	
GQT	7×7	0.7288889289		0.3733333349		0.2209523767	
	13×13	0.7883351445		0.4327795803		0.2803986370	
IOST	5×5	0.8536515995		0.4980960513		0.3457150911	
	6×6	0.8636423911		0.5080868305		0.3557058624	
	7×7	0.8699174628		0.5143619067		0.3619809556	
	8×8	0.8741142348		0.5185586841		0.3661777177	
	9×9	0.8770583374		0.5215027742		0.3691218199	
	10×10	0.5236473748		0.3712664246		0.8792029273	
		formula 1	formula 2	formula 1	formula 2	formula 1	formula 2
GQUTS	5×5	0.8888889003	0.8189709704	0.5333333421	0.4634153949	0.3809523939	0.3110344320
	6×6	0.8888888979	0.8386859193	0.5333333394	0.4831303575	0.3809523751	0.3307493955
	7×7	0.8888889008	0.8511113827	0.5333333320	0.4955558189	0.3809523895	0.3431748619
	8×8	0.8888888889	0.8594405038	0.5333333473	0.5038849433	0.3809523887	0.3515039807
	9×9	0.8888888960	0.8652927883	0.5333333366	0.5097372270	0.3809523945	0.3573562714
	10×10	0.8888888916	0.8695606956	0.5333333260	0.5140051414	0.3809523860	0.3616241943
GQUTM	14	0.8888888885	0.7979759424	0.5333333288	0.4424203913	0.3809523780	0.2900394411
	44	0.8888888823	0.8620172476	0.5333333369	0.5064616972	0.3809523803	0.3540807426
	77	0.8888888823	0.8738937178	0.5333333366	0.5183381645	0.3809523815	0.365957215
	104	0.8888889011	0.8779014912	0.5333333301	0.5223459347	0.3809523797	0.369964974
Exact Value		0.8888888		0.5333333		0.3809523	

for each quadrature of order between 1 and 30. The results are compared with the results of [26] and it is observed that the new method GQUTM is always accurate in view of both accuracy and efficiency and hence a proper balance is observed.

5. Conclusions

In continuum mechanics and in spatially discretized approximations, e.g., finite- and boundary-element methods, evaluation of integrals over arbitrary-shaped domain Ω is the important and pivotal task. Most of the integrals defy our analytical skills and we are resort to numerical integration schemes. Among all the numerical integration schemes Gaussian quadrature formulae are widely used for its simplicity and easy incorporation in computer.

In general, the Ω-shape-class is very irregular in two and three dimensional geometry. If the domain Ω is subdivided into quadrilaterals or into hexahedron respectively in two and three-dimensions, higher order Gaussian quadrature formulae are readily available. Furthermore, reduced integrations techniques compliance with the patch-test is also available [1, 4]. It is notable that there is no methodical way to design such approximate integration schemes for polygons with more than four sides. Generally simplexes e.g., triangle and tetrahedron are popular finite elements to discretize the arbitrary domain Ω. Though these are the widely used elements in FEM and BEM, Gaussian quadrature formulae for the triangular/tetrahedral domain integrals are not so developed comparing the square domain integrals. To achieve the desired accuracy of the triangular domain integral it is necessary to increase the number of points and corresponding weights. Therefore, it is an important task to make a proper balance between accuracy and efficiency of the calculations.

For the necessity of the exact evaluation of the integrals, this article shows first the integral over the

Table 5: Computed values of Example-2 for r=2, r=4, r=6.

Method	Points	Computed results of $I^{0,r}$					
		r=2		r=4		r=6	
GQT	7×7	0.7288889289		0.3733333349		0.2209523767	
	13×13	0.7883350849		0.4327795803		0.2803986370	
IOST	5×5	0.8536515995		0.4980960513		0.3457150911	
	6×6	0.8636423911		0.5080868305		0.3557058624	
	7×7	0.8699174628		0.5143619067		0.3619809556	
	8×8	0.8741142348		0.5185586841		0.3661777177	
	9×9	0.8770583374		0.5215027742		0.3691218199	
	10×10	0.8792029273		0.5236473748		0.3712664246	
		formula 1	formula 2	formula 1	formula 2	formula 1	formula 2
GQUTS	5×5	0.8189709704	0.8888889003	0.4634153949	0.5333333421	0.3110344320	0.3809523939
	6×6	0.8386859193	0.8888888979	0.4831303575	0.5333333394	0.3307493955	0.3809523751
	7×7	0.8511113827	0.8888889008	0.4955558189	0.5333333320	0.3431748619	0.3809523895
	8×8	0.8594405038	0.8888888889	0.5038849433	0.5333333473	0.3515039807	0.3809523887
	9×9	0.8652927883	0.8888888960	0.5097372270	0.5333333366	0.3573562714	0.3809523945
	10×10	0.8695606956	0.8888888916	0.5140051414	0.5333333260	0.3616241943	0.3809523860
GQUTM	14	0.7979759424	0.8888888885	0.4424203913	0.5333333288	0.2900394411	0.3809523780
	44	0.8620172476	0.8888888823	0.5064616972	0.5333333369	0.3540807426	0.3809523803
	77	0.8738937178	0.8888888823	0.5183381645	0.5333333366	0.365957215	0.3809523815
	104	0.8779014912	0.8888889011	0.5223459347	0.5333333301	0.369964974	0.3809523797
Exact Value		0.8888888		0.5333333		0.3809523	

Table 6: Computed results of Example -3

Method	Points	Computed results of $I^{0,0}$	
GQT	7×7	0.7288889289	
	13×13	0.7883350849	
IOST	5×5	0.8536515995	
	6×6	0.8636423911	
	7×7	0.8699174628	
	8×8	0.8741142348	
	9×9	0.8770583374	
	10×10	0.8792029273	
		formula 1	formula 2
GQUTS	5×5	0.8189709704	0.8888889003
	6×6	0.8386859193	0.8888888979
	7×7	0.8511113827	0.8888889008
	8×8	0.8594405038	0.8888888889
	9×9	0.8652927883	0.8888888960
	10×10	0.8695606956	0.8888888916
GQUTM	14	0.7979759424	0.8888888885
	44	0.8620172476	0.8888888823
	77	0.8738937178	0.8888888823
	90	0.8779014912	0.8888889011
Exact Value		0.8888888	

Table 7: Computed results of Example -4

Method	Points	Computed results of $I^{0,0}$	
GQT	7×7	0.02731705643	
	13×13	0.02731722965	
IOST	5×5	0.02731723353	
	6×6	0.02731723339	
	7×7	0.02731723359	
	8×8	0.02731723343	
	9×9	0.02731723344	
	10×10	0.02731723331	
		$formula1$	$formula2$
GQUTS	5×5	0.02731723329	0.02731723329
	6×6	0.02731723366	0.02731723366
	7×7	0.02731723323	0.02731723323
	8×8	0.02731723335	0.02731723335
	9×9	0.02731723349	0.02731723349
	10×10	0.02731723332	0.02731723332
GQUTM	14	0.02731722858	0.02731722858
	44	0.02731723355	0.02731723355
	77	0.02731723346	0.02731723346
	90	0.02731723357	0.02731723357
Exact Value		0.02731723349	

Table 8: The absolute error over corresponding monomials integrals

N	i	j	TP	Absolute Error
1	1	0	5	0.6531300112E-08
2	0	2	5	0.5046000631E-08
3	3	0	14	0.2096455530E-08
4	2	2	20	0.2975930894E-09
5	3	2	27	0.9426593534E-10
6	3	3	35	0.6938782665E-11
7	4	3	35	0.4528275162E-11
8	3	5	35	0.3010869684 E-11
9	3	6	44	0.6705744060E-11
10	3	7	44	0.3904583339E-11
11	4	7	44	0.6188189135E-12
12	8	4	77	0.9264116658E-12
15	7	8	65	0.4482329914E-13
26	11	15	135	0.1295867610E-17
27	13	14	135	0.7787442505E-18
28	2	26	152	0.3282154118E-18
29	0	29	152	0.3125194078E-18

triangular domain can be computed as the sum of three integrals over the square domain. In this case the readily available quadrature formulae for the square can be used for the desired accuracy. The results obtained are found accurate in view of accuracy and efficiency. Secondly, it presented new techniques to derive quadrature formulae utilizing the one dimensional Gaussian quadrature formulae and that overcomes all the difficulties pertinent to the higher order formulae. The first technique (GQUTS) derives $m \times m$ point quadrature formula utilizing the one dimensional m-point Gaussian quadrature formula. Finally, in the second technique (GQUTM) $\frac{m(m+1)}{2} - 1$ point quadrature formula is derived utilizing the m-point one dimensional Gaussian quadrature formula. It is observed that this scheme is appropriate for the triangular domain integrals as it requires less computational effort for desired accuracy. Through practical application examples, it is demonstrated that the new appropriate Gaussian quadrature formula for triangles are accurate in view of accuracy and efficiency and hence a proper balance is observed.

Thus, we believe that the newly derived appropriate quadrature formulae for triangles will ensure the exact evaluation of the integrals in an efficient manner and enhance the further utilization of triangular elements for numerical solution of field problems in science and engineering.

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