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The immersed interface method using a finite element formulation ☆

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Abstract

A finite element method is proposed for one dimensional interface problems involving discontinuities in the coefficients of the differential equations and the derivatives of the solutions. The interfaces do not have to be one of grid points. The idea is to construct basis functions which satisfy the interface jump conditions. By constructing an interpolating function of the solution, we are able to give a rigorous error analysis which shows that the approximate solution obtained from the finite element method is second order accurate in the infinity norm. Numerical examples are also provided to support the method and the theoretical analysis. Several numerical approaches are also proposed for dealing with two dimensional problems involving interfaces.

Keywords: Immersed interface method; Finite element method; Modified basis function; Natural jump condition; Error analysis

1. Introduction

Consider the following model problem:

$$-(\beta(x)\,u')' + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1, \tag{1.1}$$

$$u(0) = 0, \qquad u(1) = 0. \tag{1.2}$$

We assume that $0 < \beta(x)$ is piecewise continuous and may have finite jumps at interfaces α_1, α_2, $\ldots$, α_s. Across the interfaces, sometimes also called internal boundaries, the natural jump conditions hold:

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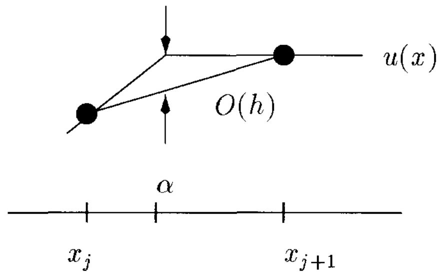


Fig. 1. A diagram shows why a finite element method cannot be second order accurate in the infinity norm if the interface is not a grid point.

$$[u]_{\alpha_i} \stackrel{\text{def}}{=} \lim_{x \to \alpha_i^+} u(x) - \lim_{x \to \alpha_i^-} u(x) = 0, \tag{1.3}$$

$$[\beta u_x]_{\alpha_i} \stackrel{\text{def}}{=} \lim_{x \to \alpha_i^+} \beta(x)\, u'(x) - \lim_{x \to \alpha_i^-} \beta(x)\, u'(x) = 0. \tag{1.4}$$

The solution of (1.1) is typically non-smooth at the interfaces if $\beta(x)$ has a finite jump at each interface.

The problem can be solved by both finite difference methods and finite element methods. The immersed interface method (IIM) [6,7,9–11] is an efficient finite difference approach for interface problems with discontinuities and singularities. The solution obtained from the IIM is typically second order accurate in the infinity norm regardless of the relative position between the grid points and the interfaces. However, for two or higher dimensional problems, the resulting linear system obtained from the IIM may not be symmetric positive definite.

If the finite element method with the standard linear basis is used for (1.1) with presence of interfaces, second order accurate solutions can still be obtained if the interfaces lie on the grid points. This can be proved strictly in one dimensional space. For higher dimensional problems, the analysis is usually given in an integral norm which is weaker than the infinity norm, see [1,3,4,13], etc. If any of the interfaces is not a grid point, then the solution obtained from the finite element method is only first order accurate in the infinity norm, see Fig. 1. For two or higher dimensional problems, it is difficult and costly to construct a body-fitting grid so that the interface aligns with the triangulation, especially for moving interface problems.

In this paper, we try to develop a numerical method which maintains the advantages of the simple grid structure of the finite difference method and the nice theoretical properties of the finite element method. The idea is to take a simple Cartesian grid, for example, a uniform grid, and modify the basis functions so that the interface jump relations are satisfied. With the simple grid, or triangulation, the finite element method corresponds to a finite difference method in which the resulting linear system of equations is symmetric positive definite. By choosing modified basis functions, second order accuracy is achieved in the infinity norm for one dimensional problems. Inhomogeneous jump conditions then can be taken care of easily by adding some correction terms according to the immersed interface method. We also propose some numerical methods for constructing basis functions for two dimensional problems involving interfaces. While second order convergence is preserved in the energy norm for those methods, the convergence of those methods in the infinity norm is still under investigation.

2. Modification of the linear basis

Define the standard bilinear form

$$a(u,v)=\int_0^1 \big(\beta(x)u'(x)v'(x)+q(x)u(x)v(x)\big)\,\mathrm{d}x,\quad u(x),v(x)\in H_0^1(0,1), \tag{2.5}$$

where $H_0^1(0,1)$ is the Sobolev space. The solution of the differential equation $u(x)\in H_0^1(0,1)$ is also the solution of the following variational problem:

$$a(u,v)=(f,v)=\int_0^1 f(x)v(x)\,\mathrm{d}x,\quad \forall v\in H_0^1(0,1). \tag{2.6}$$

Without loss of generality, we assume that there is only one interface α in the interval $(0,1)$. Integration by parts over the separated intervals $(0,\alpha)$ and $(\alpha,1)$ yields

$$0=\int_0^\alpha \{-(\beta u')'+qu-f\}v+\beta^-u_x^-v^- \tag{2.7}$$

$$+\int_\alpha^1 \{-(\beta u')'+qu-f\}v-\beta^+u_x^+v^+. \tag{2.8}$$

The superscripts $-$ and $+$ indicate the limiting value as x approaches α from the left and right, respectively, and $u_x=u'$. Recall that $v^-=v^+$ for any v in H_0^1, it follows that the differential equation holds in each interval and that

$$[u]=u^+-u^-=0,\qquad [\beta u_x]=\beta^+u_x^+-\beta^-u_x^-=0,$$

where we have dropped the subscript α in the jumps since there is only one interface. These relations are the same as in (1.3), (1.4), which indicates that the discontinuity in the coefficient $\beta(x)$ does not cause any trouble for the theoretical analysis of the FEM and the weak solution will satisfy the jump conditions (1.3), (1.4).

Now let us turn our attention to the numerics. For simplicity, we use a uniform grid $x_i=ih$, $i=0,1,\ldots,N$, with $x_0=0$, $x_N=1$ and $h=1/N$ in our discussion. The standard linear basis function satisfies

$$\phi_i(x_k)=\begin{cases}1, & \text{if } i=k,\\ 0, & \text{otherwise.}\end{cases} \tag{2.9}$$

The solution $u_h(x)$ is a specific linear combination of the basis function from the finite dimensional space V_h:

$$V_h=\left\{v_h\colon\ v_h=\sum_{i=1}^{N-1}\eta_i\phi_i(x)\right\}, \tag{2.10}$$

$$a(u_h,v_h)=(f,v_h),\quad \text{for } \forall v_h\in V_h. \tag{2.11}$$

If an interface is not one of grid points x_i, usually the solution u_h is only first order accurate in the infinite norm, see Fig. 1. The problem is that some basis functions which have non-zero support near the interface do not satisfy the natural jump condition (1.4) at the interface.

The solution is to modify the basis functions in such a way that natural jump conditions are satisfied:

$$\phi_i(x_k) = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{otherwise,} \end{cases} \tag{2.12}$$

$$[\phi_i] = 0, \tag{2.13}$$

$$[\beta\phi_i'] = 0. \tag{2.14}$$

Obviously, if $x_j < \alpha < x_{j+1}$, then only ϕ_j and ϕ_{j+1} need to be changed to satisfy the second jump condition. Using an undetermined coefficient method, we can conclude that

$$\phi_j(x) = \begin{cases} 0, & 0 \leqslant x < x_{j-1}, \\ \dfrac{x - x_{j-1}}{h}, & x_{j-1} \leqslant x < x_j, \\ \dfrac{x_j - x}{D} + 1, & x_j \leqslant x < \alpha, \\ \dfrac{\rho\,(x_{j+1} - x)}{D}, & \alpha \leqslant x < x_{j+1}, \\ 0, & x_{j+1} \leqslant x \leqslant 1, \end{cases} \tag{2.15}$$

where

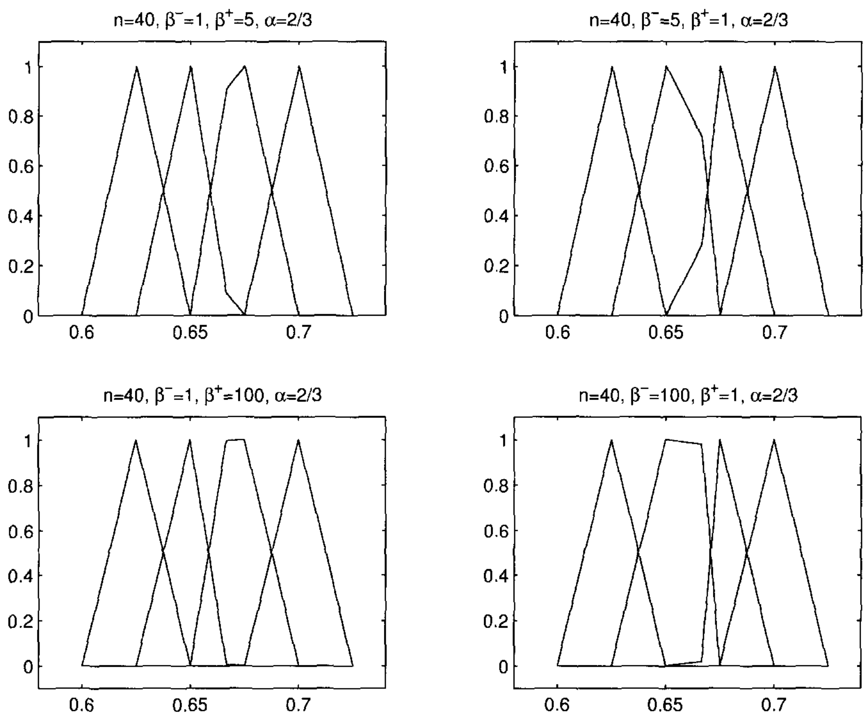


Fig. 2. Plot of some basis function near the interface with different β^- and β^+.

$$\rho = \frac{\beta^-}{\beta^+}, \qquad D = h - \frac{\beta^+ - \beta^-}{\beta^+}(x_{j+1} - \alpha), \tag{2.16}$$

and

$$\phi_{j+1}(x) = \begin{cases} 0, & 0 \leqslant x < x_j, \\ \dfrac{x - x_j}{D}, & x_j \leqslant x < \alpha, \\ \dfrac{\rho(x - x_{j+1})}{D} + 1, & \alpha \leqslant x < x_{j+1}, \\ \dfrac{x_{j+2} - x}{h}, & x_{j+1} \leqslant x \leqslant x_{j+2}, \\ 0, & x_{j+2} \leqslant x \leqslant 1. \end{cases} \tag{2.17}$$

Fig. 2 shows several plots of the modified basis functions $\phi_j(x)$, $\phi_{j+1}(x)$, and some neighboring basis functions, that are the standard hat functions. At the interface, we can see clearly the kink in the basis function which reflect the natural jump conditions.

3. The theoretical analysis

In this section, we prove that the solution obtained from the finite element method with the modified basis function is second order accurate in the infinite norm.

For the sake of clean and concise proof, we derive the theoretical analysis for the simple model:

$$-\beta(x)u'' = f(x), \quad f(x) \in C[0,1], \qquad 0 \leqslant x \leqslant 1, \tag{3.18}$$

$$u(0) = 0, \qquad u(1) = 0, \tag{3.19}$$

with

$$\beta(x) = \begin{cases} \beta^-, & \text{if } 0 \leqslant x < \alpha, \\ \beta^+, & \text{if } \alpha < x \leqslant 1, \end{cases}$$

where β^- and β^+ are two constants. The solution $u(x) \in H_0^1$ satisfies the natural jump conditions at α. If the value of the solution at α is known, say u_α, then the problem is equivalent to the following two separated problems:

$$\begin{cases} -\beta^- u'' = f(x), & 0 \leqslant x < \alpha, \\ u(0) = 0, & u(\alpha) = u_\alpha, \end{cases} \quad \text{and} \quad \begin{cases} -\beta^+ u'' = f(x), & \alpha < x \leqslant 1, \\ u(\alpha) = u_\alpha, & u(1) = 0. \end{cases}$$

Therefore from the regularity theory we know that $u(x) \in C^2[0,1]$ in each sub-domain and $u_{xx}^- = \lim_{x\to\alpha^-} u''(x)$ and $u_{xx}^+ = \lim_{x\to\alpha^+} u''(x)$ are finite. We define

$$\|u''\|_\infty = \max\left\{|u_{xx}^-|, |u_{xx}^+|, \sup_{0<x<\alpha} |u_{xx}|, \sup_{\alpha<x<1} |u_{xx}|\right\}, \tag{3.20}$$

which is bounded.

3.1. The interpolant of the solution

As in the standard FEM analysis, an interpolating function of the solution plays an important role in the error analysis. In this subsection, we will define the piecewise linear function in the space V_h which also interpolates $u(x)$ at the node points. Assuming that $x_j \leqslant \alpha < x_{j+1}$, we define an interpolant of $u(x)$ as follows:

$$u_I(x) = \begin{cases} \dfrac{x_{i+1} - x}{h} u(x_i) + \dfrac{x - x_i}{h} u(x_{i+1}), & i \neq j,\ x_i \leqslant x \leqslant x_{i+1}, \\ u(x_j) + \kappa(x - u(x_j)), & x_j \leqslant x < \alpha, \\ u(x_{j+1}) + \kappa\,\rho(x - x_{j+1}), & \alpha \leqslant x \leqslant x_{j+1}, \end{cases} \tag{3.21}$$

where

$$\rho = \frac{\beta^-}{\beta^+}, \qquad \kappa = \frac{u(x_{j+1}) - u(x_j)}{\alpha - x_j - \rho(\alpha - x_{j+1})}. \tag{3.22}$$

It is easy to verify that

$$u_I(x_i) = u(x_i), \quad i = 0, 1, \ldots, N-1, \tag{3.23}$$

$$[u_I] = 0, \qquad [\beta u_I'] = 0, \tag{3.24}$$

and hence $u_I(x) \in V_h$. Before giving an error bound for $\|u_I(x) - u(x)\|_\infty$, we need the following lemma which gives the error estimates for the first derivatives of $u_I(x)$ approximating $u'(x)$.

Lemma 3.1. *Given* $u_I(x)$ *as defined in* (3.21), *the following inequalities hold*:

$$\alpha - x_j - \rho(\alpha - x_{j+1}) \geqslant \min\{\tfrac{1}{2}h,\ \tfrac{1}{2}h\,\rho\}, \tag{3.25}$$

$$|\kappa - u_x^-| \leqslant C\,\|u''\|_\infty\, h, \tag{3.26}$$

$$|\rho\,\kappa - u_x^+| \leqslant C\,\|u''\|_\infty\, h, \tag{3.27}$$

where

$$C = \frac{2\max\{1,\ \rho\}}{\min\{1,\ \rho\}}. \tag{3.28}$$

So C acts like a condition number for the interface problem.

Proof. It is obvious that

$$\begin{cases} \alpha - x_j - \rho(\alpha - x_{j+1}) \geqslant \rho(x_{j+1} - \alpha) \geqslant \tfrac{1}{2}h\rho, & \text{if } \alpha - x_j < \tfrac{1}{2}h, \\ \alpha - x_j - \rho(\alpha - x_{j+1}) \geqslant \alpha - x_j \geqslant \tfrac{1}{2}h, & \text{if } \alpha - x_j > \tfrac{1}{2}h, \end{cases}$$

which concludes the first inequality. Using the Taylor expansion about α, we have

$$\begin{aligned} |\kappa - u_x^-| &= \left| \frac{u(x_{j+1}) - u(x_j)}{\alpha - x_j - \rho(\alpha - x_{j+1})} - u_x^- \right| \\ &= \left| \frac{u^+ + u_x^+(x_{j+1} - \alpha) + \frac{1}{2}u_{xx}(\xi_1)(x_{j+1} - \alpha)^2}{\alpha - x_j - \rho(\alpha - x_{j+1})} \right. \\ &\qquad \left. - \frac{u^- + u_x^+(x_j - \alpha) + \frac{1}{2}u_{xx}(\xi_2)(x_j - \alpha)^2}{\alpha - x_j - \rho(\alpha - x_{j+1})} - u_x^- \right|, \end{aligned}$$

where $\xi_1 \in (\alpha, x_{j+1})$ and $\xi_2 \in (x_j, \alpha)$. With the jump conditions $u^+ = u^-$ and $u_x^+ = \rho u_x^-$, the expression above is simplified to

$$|\kappa - u_x^-| = \frac{\left|\frac{1}{2}u_{xx}(\xi_2)(x_{j+1}-\alpha)^2\right| + \left|\frac{1}{2}u_{xx}(\xi_2)(x_j-\alpha)^2\right|}{\alpha - x_j - \rho(\alpha - x_{j+1})} \leqslant \frac{1}{\min\left\{\frac{1}{2},\, \frac{1}{2}\rho\right\}} \|u''\|_\infty h,$$

which implies the second inequality (3.26). At last,

$$|\rho\kappa - u_x^+| = \rho|\kappa - u_x^-| \leqslant C\|u''\|_\infty h.$$

This completes the proof of the lemma. □

We are now ready to prove the following theorem on the accuracy of the interpolating function $u_I(x)$.

Theorem 3.2. *If $u_I(x)$ is given as in* (3.21), *then*

$$\big\|(x) - u_I(x)\big\|_\infty \leqslant \overline{C}h^2\|u''\|_\infty, \tag{3.29}$$

where

$$\overline{C} = \frac{2\max\{1,\, \rho\}}{\min\{1,\, \rho\}} + \frac{3}{2} = C + \frac{3}{2}.$$

Proof. Again we assume that $\alpha \in [x_j, x_{j+1})$ for some integer $0 < j \leqslant N-1$. For any $x \in [x_i, x_{i+1}]$ which does not contain the interface α, from the standard interpolation theory, we know that

$$\big|u(x) - u_I(x)\big| \leqslant \tfrac{1}{8}h^2\|u''\|_\infty \leqslant \overline{C}h^2\|u''\|_\infty.$$

If $x_j \leqslant x \leqslant \alpha$, then

$$\begin{aligned} u(x) &= u(x_j) + u'(x_j)(x - x_j) + \tfrac{1}{2}(x - x_j)^2 u''(\xi_1) \\ &= u(x_j) + u_x^-(x - x_j) + u''(\xi_2)(x - \alpha)(x - x_j) + \tfrac{1}{2}(x - x_j)^2 u''(\xi_1), \end{aligned}$$

where $\xi_1 \in (x_j, \alpha)$ and $\xi_2 \in (x, \alpha)$ from the intermediate value theorem, and $u_x^- = \lim_{x\to\alpha^-} u'(x)$. Thus using the bound in (3.26), and the fact that $|x - \alpha| \leqslant h$ and $|x - x_j| \leqslant h$, we have

$$\begin{aligned} \big|u(x) - u_I(x)\big| &= \big|(u_x^- - \kappa)(x - x_j)\big| + \big|u''(\xi_2)(x - \alpha)(x - x_j) + \tfrac{1}{2}(x - x_j)^2 u''(\xi_1)\big| \\ &\leqslant Ch^2\|u''\|_\infty + \tfrac{3}{2}h^2\|u''\|_\infty \leqslant \overline{C}h^2\|u''\|_\infty. \end{aligned}$$

The proof is similar if $\alpha < x \leqslant x_{j+1}$. □

3.2. Convergence theorem for the finite element method

We are now ready to prove that the approximate solution obtained from the FEM with the modified linear basis is second order accurate to the exact solution in the infinite norm. First, we need to prove the following lemma.

Lemma 3.3. *If $u(x)$ is the solution of* (1.1), (1.2), *and $u_I(x)$ is the interpolating function defined in* (3.21), *then*

$$\int_{x_i}^{x_{i+1}} \beta(x)\big(u(x) - u_I(x)\big)' v_h'(x)\,\mathrm{d}x = 0, \quad \forall v_h \in V_h, \quad \textit{and} \quad 0 \leqslant i \leqslant N-1. \tag{3.30}$$

Proof. If $\alpha \notin [x_i, x_{i+1})$, then $\beta(x)$ and $v_h'(x)$ are two constants. So

$$\int_{x_i}^{x_{i+1}} \beta(x)(u(x) - u_I(x))' v_h'(x)\,\mathrm{d}x = \beta(x_{i+1/2})\, v_h'(x_{i+1/2}) \int_{x_i}^{x_{i+1}} \big(u(x) - u_I(x)\big)'\,\mathrm{d}x$$
$$= \beta(x_{i+1/2})\, v_h'(x_{i+1/2}) \big(u(x) - u_I(x)\big)\Big|_{x_i}^{x_{i+1}} = 0,$$

where $x_{i+1/2} = x_i + h/2$. On the other hand if $\alpha \in [x_j, x_{j+1})$, then

$$\int_{x_j}^{x_{j+1}} \beta(x)\big(u(x) - u_I(x)\big)' v_h'(x)\,\mathrm{d}x$$
$$= \beta^- v_h'(\xi_1) \int_{x_j}^{\alpha} \big(u(x) - u_I(x)\big)'\,\mathrm{d}x + \beta^+ v_h'(\xi_2) \int_{\alpha}^{x_{j+1}} \big(u(x) - u_I(x)\big)'\,\mathrm{d}x$$
$$= \beta^- v_h'(\xi_1)(u^- - u_I^-) - \beta^+ v_h'(\xi_2)(u^+ - u_I^+) = 0,$$

where ξ_1 and ξ_2 are any two points in the interval (x_j, α) and (α, x_{j+1}), respectively. In the derivation above, we have used the natural jump condition (1.3), (1.4) for the basis function v_h and continuity conditions for $u(x)$ and $u_I(x)$ at α:

$$\beta^- v_h'(\xi_1) = \beta^+ v_h'(\xi_2), \qquad u^+ = u^-, \qquad u_I^+ = u_I^-. \qquad \square$$

Below is the main theorem of convergence for the modified finite element method.

Theorem 3.4. *Let $u_h(x)$ be the solution obtained from the finite element method with the modified basis function. Then*

$$\big\|u(x) - u_h(x)\big\|_\infty \leqslant C\|u''\|_\infty h^2, \tag{3.31}$$

where

$$C = \frac{2\max\{1,\, \rho\}}{\min\{1,\, \rho\}} + \frac{3}{2}. \tag{3.32}$$

Proof. For any $v_h \in V_h$, we have

$$a(u - u_I, v_h) = a(u - u_h + u_h - u_I, v_h) = a(u - u_h, v_h) + a(u_h - u_I, v_h) = a(u_h - u_I, v_h).$$

From the definition of $a(u, v)$ and Lemma 3.3, we know that

$$a(u - u_I, v_h) = \int_0^1 \beta(x)(u - u_I)' v_h'\,\mathrm{d}x = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \beta(x)\big(u(x) - u_I(x)\big)' v_h'(x)\,\mathrm{d}x = 0.$$

Take $v_h = u_h - u_I \in V_h$, we conclude $a(u_h - u_I, u_h - u_I) = 0$, which implies that $u_h(x) \equiv u_I(x)$. Thus

$$\big|u(x) - u_h(x)\big| \leqslant \big|u(x) - u_I(x)\big| + \big|u_I(x) - u_h(x)\big| \leqslant \big|u(x) - u_I(x)\big| \leqslant C\|u''\|_\infty h^2. \qquad \square$$

Remark 3.1. The conclusion can be easily extended to the more general case (1.1), (1.2) with variable $\beta(x)$. The key modification is the following:

$$a(u-u_I,v_h)=\sum_{i=0,i\neq j,j+1}^{N-1}\int_{x_i}^{x_{i+1}}\left(\beta(x)-\overline{\beta}_{i+1/2}\right)(u-u_I)'v_h'\,\mathrm{d}x$$

$$+\int_{x_j}^{\alpha}\left(\beta(x)-\overline{\beta}_l\right)(u-u_I)'v_h'\,\mathrm{d}x+\int_{\alpha}^{x_{i+1}}\left(\beta(x)-\overline{\beta}_r\right)(u-u_I)'v_h'\,\mathrm{d}x,$$

where $\overline{\beta}_{i+1/2}$, $\overline{\beta}_l$ and $\overline{\beta}_r$ are the average values of $\beta(x)$ in the intervals $[x_i,x_{i+1}]$, $[x_j,\alpha]$ and $[\alpha,x_{j+1}]$, respectively, which are first order approximations to $\beta(x)$ in that specific interval. Note that $u_I(x)$ is a second order approximation to $u(x)$ in the infinite norm, which means that $u_I'(x)$ is a first order approximation to $u'(x)$ except at grid points and the interface α. Thus we have

$$a(u_h-u_I,v_h)\leqslant C_1h\|u-u_I\|_1\|v_h\|_1\leqslant C_1h^2\|u''\|_\infty\|v_h\|_1.$$

Taking $v_h=u_h-u_I\in V_h$, we conclude

$$\|u_h-u_I\|_1\leqslant C_1h^2\|u''\|_\infty.$$

The final inequality then follows:

$$\begin{aligned}\left|u(x)-u_h(x)\right|&\leqslant\left|u(x)-u_I(x)\right|+\left|u_I(x)-u_h(x)\right|\\&\leqslant\left|u(x)-u_I(x)\right|+\left\|u_h(x)-u_I(x)\right\|_1\leqslant\widehat{C}\|u''\|_\infty h^2,\end{aligned}$$

with a different error constant $\widehat{C}$.

4. Numerical examples

We have done quite a number of numerical tests. All the results agree with the theoretical analysis. The integrals

$$\int_{x_i}^{x_{i+1}}\phi_i(x)f(x)\,\mathrm{d}x\quad\text{and}\quad\int_{x_i}^{x_{i+1}}\phi_i'(x)\phi_j'(x)\,\mathrm{d}x$$

are evaluated using the trapezoidal rule. We just present one example below.

The differential equation is

$$(\beta u_x)_x=12x^2,\qquad 0\leqslant x\leqslant 1,\quad \beta=\begin{cases}\beta^- & \text{if } x<\alpha,\\ \beta^+ & \text{if } x>\alpha,\end{cases}$$

$$u(0)=0,\qquad u(1)=1/\beta^+ + \left(1/\beta^- - 1/\beta^+\right)\alpha^4.$$

The natural jump conditions (1.3), (1.4) are satisfied. The exact solution is

$$u(x)=\begin{cases} x^4/\beta^- & \text{if } x<\alpha,\\ x^4/\beta^+ + (1/\beta^- - 1/\beta^+)\alpha^4 & \text{if } x>\alpha,\end{cases}$$

where the parameters β^- and β^+ are two constants.

Since Simpson's rule has degree of precision "three" and $f(x)$ is a quadratic, there are no errors in computing $\int \phi_i(x) f(x)\,\mathrm{d}x$ and $\int \phi_i'(x)\phi_j'(x)\,\mathrm{d}x$. We expect the solution obtained using the FEM method to be the same as the interpolating function defined in (3.21), provide there is no round-off error involved. In other words, the computed solution agrees with the exact solution at grid points and is second order accurate at any other points. Numerical experiments have confirmed the theoretical analysis. The infinity norm of the computed solution at grid points is between 6×10^{-15} to 3×10^{-13} in double precision. At any other points, the FEM solution is defined as

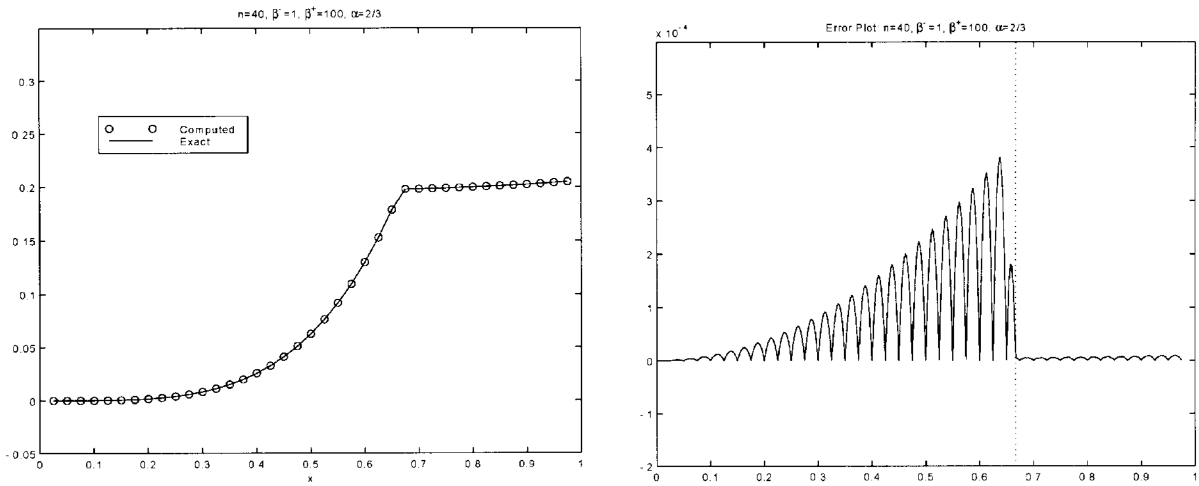
$$u_h(x)=\sum_{i=0}^{N} u_i\phi_i(x), \tag{4.33}$$

where u_i is the computed solution at the grid point x_i, the error decreases by a factor of 4 if we double the grid size. Table 1 shows the grid refinement analysis in the infinity norm for two different points which are not part of the grid. In the first case, $\beta^-=1$, $\beta^+=100$, and the interface is $\alpha=\frac{2}{3}$ which is not a grid point. We see that the computed solution at the interface itself has average second order accuracy. In the second case, we take the same β^- and β^+, but the interface is $\alpha=0.5$ which is a grid point. Since there is no interface between the grid points, the solution at the interface α is again accurate to the machine precision up to a factor of the condition number of the discrete linear system. The right part of the table shows the grid refinement analysis at $\alpha+\frac{2}{3}$ which is not a grid point. We see that the error is reduced by a factor of 4. Notice that, for interface problems, the error constant which is O(1) may not approach to a constant. It will depend on the relative position of the interface and the grid. This is the case in the left part of the table. By the second order accuracy, we actually mean the average convergence rate of the solution, the reader is referred to [10,12] for more information on the error analysis. For the second case, since the interface is a grid point, the error constant will indeed approach to a fixed number.

Fig. 3(a) is the plot of the solution with 40 grid points. There is no difference between the computed and the exact solution at the grid points. The differences in other places are too small to be visible.

Table 1
Grid refinement analysis for the example with $\beta^-=1$, $\beta^+=100$. The left part of the table: the error of the solution evaluated at the interface $x=\alpha=\frac{2}{3}$. The right part of the table: the error of the solution evaluated at $x=0.5+\frac{1}{3}$, $\alpha=0.5$

n	e_n	e_n/e_{2n}	e_n/e_{4n}	n	e_n	e_n/e_{2n}
20	4.4312×10^{-5}			20	2.2844×10^{-5}	
40	5.4822×10^{-6}	8.0829		40	5.8259×10^{-6}	3.9211
80	2.7347×10^{-6}	2.0047	16.2038	80	1.4420×10^{-6}	4.0403
160	3.4478×10^{-7}	7.9318	15.9010	160	3.6229×10^{-7}	3.9801
320	1.7038×10^{-7}	2.0235	16.0503	320	9.0347×10^{-8}	4.0100
640	2.1582×10^{-8}	7.8948	15.9752	640	2.2615×10^{-8}	3.9950



(a) (b)

Fig. 3. Comparison of the computed solution and exact solution when $n = 40$. (a) The solution plot. (b) The error plot.

Fig. 3(b) is the error plot of the error in the entire interval. We see the errors are zero at grid points and $O(h^2)$ at other points.

5. Constructing basis functions in two dimensions

The model equation in two dimensions on a rectangular region with a closed interface is

$$\nabla \cdot (\beta \nabla u) = f(x, y), \quad (x, y) \in \Omega, \tag{5.34}$$

$$\text{given BC on } \partial\Omega, \tag{5.35}$$

see the diagram in Fig. 4. The natural jump conditions across the interface Γ are

$$[u] = 0, \qquad [\beta u_n] = 0, \tag{5.36}$$

where u_n is the normal derivative.

As discussed in previous sections, we will use a uniform triangulation, see Fig. 5. If a cell contains no interface, we can use the standard linear basis function over that cell. It is more difficult to construct basic function in two dimensions when interface cuts through the uniform triangulation. It is true that we can easily find piecewise linear, or quadratic, or cubic function which interpolates the solution of (5.34), (5.35) to second or higher order accuracy using the Taylor expansion. The difficulty in constructing the basis function is the requirements of continuity in the entire region and the jump conditions across the interface. There are several approaches currently under investigation. Bube and Kaupe [2] are tying to use a quadrilateral triangulation. Hou and Wu [5] are experimenting with both conforming and non-conforming basis functions. Below we propose an approach which is in the same spirit as our discussion for one dimensional problem. We will stick with conforming basis functions, that is, the basis functions belong to $H_0^1(\Omega)$.

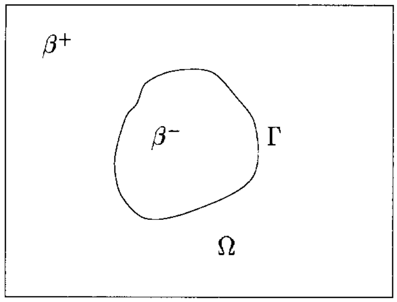


Fig. 4. A diagram for a model problem in two dimensions.

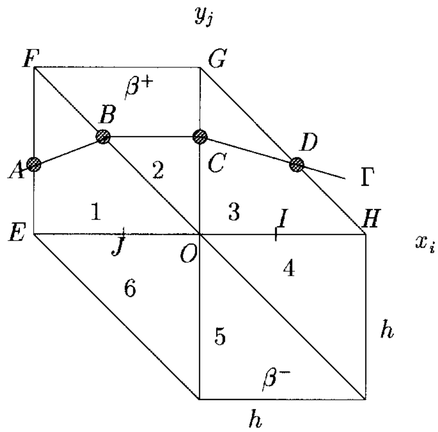


Fig. 5. A typical cell with the interface cutting through.

5.1. A coupled approach

Now take a typical case as shown in Fig. 5. An arbitrary closed interface can be approximated by piecewise line segments. We want to find a basis function $\phi(x, y)$, for example, centered at (x_i, y_j) which satisfies

$$\phi(x, y) \in C(\Omega), \qquad \phi(x_i, y_j) = 1, \tag{5.37}$$

$$[\phi] = 0, \qquad [\beta\phi_n] = 0. \tag{5.38}$$

It is quite obvious that a linear basis function will not work. Instead, we try to use a piecewise quadratic basis function. In Fig. 5, the interface does not cut through the triangles 4, 5 and 6. Therefore the basis function can be taken as the standard linear basis function over those triangles.

The triangles 1, 2 and 3 contain a portion of the interface which divides the region into six pieces, three triangles and three quadrilaterals. The piecewise quadratic function over the six pieces can be determined using the *undetermined coefficient method.* The total number of degrees of freedom is 36 without any constrains. The continuity constrains involving both the sides of the triangles and the interface are 33. The remaining 3 degrees of freedom are then used to satisfy the flux jump condition $[\beta\phi_n] = 0$ to certain degree. For instance, we can force $[\beta\phi_n]$ to be zero at the mid-points of linear segments $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$. At other points of the interface, $[\beta\phi_n]$ will also be very small from the continuity condition of the solution and the assumption that the normal direction of the interface does not change too much in the cell.

In the end, we need to set up a linear system of equation for the coefficients of the quadratic basis function. The number of unknowns can be greatly reduced if we take advantage of the boundary conditions. For example, the function $\phi(x,y)$ on the quadrilateral $ABOE$ can be written as

$$\phi(x,y) = a_0 + a_1(x - x_{i-1}) + a_2(y - y_j) + a_3(x - x_{i-1})(y - y_j).$$

It is not so easy to implement the approach discussed above because the basis functions on several cells are coupled together. This is the price which we need to pay for interface problems to obtain more accurate results. However we can simplify the process of constructing the basis function if we can *pre-determine* the values of the basis function at points B, and C as in the example of Fig. 5. An interpolation approach to determine those values will be discussed later in this section.

5.2. *A decoupled approach*

Suppose we can pre-determine the values of the basis function at those points B, and C in Fig. 5, then we can construct a bilinear function in each quadrilateral which interpolates the function values at the vertices and those intersections such as the points B, and C in Fig. 5. The bilinear functions $\phi(x,y)$ will be linear on the boundary of the region as shown in Fig. 5. For example, the bilinear function $\phi(x,y)$ on the quadrilateral $ABOE$ is

$$\phi(x,y) = \frac{x - x_{i-1}}{h} + \frac{y - y_j}{h} + 1 + q\,(x - x_{i-1})(y - y_j),$$

where q is chosen such that $\phi(x_B, y_B) = \phi_B$, the pre-determined value of the basis function at B.

Once we have determined the bilinear function on each quadrilateral, then we know the values of the basis function at mid-points of all sides of each triangles. Thus a quadratic function is easily determined from the six values of the basis function on each triangle.

In the approach we described above, we are almost able to find the basis function on each triangle separately, which makes it easier to assemble the stiffness matrix.

5.3. *Interpolation scheme for the pre-determined values*

The approach described above rely on the values at intersections of the interface and the interior sides of the triangles. Ideally, the basis function can be taken as the solution of the following Poisson equation with natural jump conditions:

$$\nabla \cdot (\beta \nabla \phi) = 0, \tag{5.39}$$

$$[\phi] = 0, \qquad [\beta\phi_n] = 0. \tag{5.40}$$

The boundary condition is

$$\phi = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{on } EO, \\ 1 - \dfrac{x - x_i}{h} & \text{on } EF, \\ 0 & \text{on } EG,\ GH \text{ and } HF. \end{cases} \tag{5.41}$$

Once we know the solution of the PDE above, we can get those pre-determined values. However, it is too costly to solve the PDE above at all the cells which contain the interface. What we can do, however, is to use an interpolation scheme in terms of the boundary values, the jump conditions, and the partial differential equation itself, for example,

$$\phi_B = \alpha_1\phi_E + \alpha_2\phi_A + \alpha_3\phi_F + \alpha_4\phi_G + \alpha_5\phi_D + \alpha_6\phi_H + \alpha_7\phi_I + \alpha_8\phi_O + \alpha_9\phi_J.$$

The coefficients can be determined using the weighted least squares interpolation [10] and the procedure is briefly described below:

- Select a point $\vec{X}^*$ on the interface.
- Use the local coordinates at $\vec{X}^*$ in the tangential and normal directions of the interface.
- Use the Taylor expansion over $\vec{X}^*$ to expand the values from each side of the interface.
- Eliminate the quantities of one side, such as the solution, the derivatives up to second order, in terms of another using the jump conditions and the differential equation.
- Set up and solve the linear system of equation to get the coefficients of the interpolation.

The details can be found in [8,10] with some modification.

Theoretically, if the basis functions belong to $H_0^1(\Omega)$ space and satisfy the natural jump conditions, then the standard error analysis using the energy norm would apply. So we would have the standard convergence result even if there is an interface in the solution domain. It is not easy to see or prove whether the methods described above are still second order accurate in the infinity norm. However, the approaches described above certainly has better accuracy compared to the straightforward finite element method with no modifications. The price is the extra cost at those cells where the interface cuts through.

In summary, the modified finite element method using the simple or uniform triangulation is very accurate for one-dimensional problems and very promising for two-dimensional problems. The corresponding finite difference method would allow us to deal with inhomogeneous jump conditions. However, the analysis for two or higher dimensional interface problems is far from complete. We hope the ideas presented in this paper will eventually lead to the development of some efficient finite element methods with simple triangulations for two and three dimensional problems.

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