MA 584
Homework #3, Part A
Due November 17, 2020

1. (a): State and prove the continuous and discrete maximum principle for 1D elliptic problem.

   (b): Use the result to show that the finite difference solution using the central finite difference scheme is second order accurate in the infinity norm.

2. Derive the fourth-order compact finite difference scheme for $u''(x) = f(x)$, $a < x < b$ with $u(a) = u_a$, $u(b) = u_b$.

3. (a) Give a finite difference formula to approximate $\dfrac{\partial^3 u}{\partial x^2 \partial y}$ (or u_{xxy}) for a function $u(x, y)$ assuming that the step sizes are h_x and h_y.

   (b) (True or False) If a numerical method is stable and consistent, then the method converges.

   (c) (True or False) If a finite difference method is convergent, then the method has to be stable and consistent.

   (d) (True or False) A finite difference scheme converges for a particular problem, it has to be consistent.

4. (Programming Part) Implement and compare the Gauss-Seidel, and the SOR (trying to find the best ω by testing), methods for the following elliptic problem:

$$u_{xx} + p(x, y)u_{yy} + r(x, y)u(x, y) = f(x, y)$$
$$a < x < b; \qquad c < y < d,$$

with the following boundary conditions:

$$u(a, y) = 0, \quad u(x, c) = 0, \quad u(x, d) = 0, \quad \frac{\partial u}{\partial x}(b, y) = -\pi \sin(\pi y).$$

Test and debug your code for the case $\quad 0 \le x, y \le 1$, and

$$p(x, y) = (1 + x^2 + y^2), \qquad r(x, y) = -xy.$$

The the source term $f(x, y)$ is determined from the exact solution

$$u(x, y) = \sin(\pi x)\sin(\pi y).$$

Do the grid refinement analysis for $n = 16$, $n = 32$, and $n = 64$ (if possible) in the infinity norm (**Hint:** In matlab, use $max(max(abs(e)))$). Take the tolerance as 10^{-5}. Compare also the number of iterations and test the optimal relaxation factor ω. Plot the solution and the error for $n = 32$.

Having made sure that you code is working correctly, try your code with a point source $f(x, y) = \delta(x - 0.5)\delta(y - 0.5)$ and $u_x = -1$ at $x = 1$, with $p(x, y) = 1$ and $r(x, y) = 0$. Note that the $u(x, y)$ can be interpreted as the steady state temperature distribution of a room with insulated wall on three sides, a constant heat flow in from one side, and a point source (a heater for example) in the room. Note that the heat source can be expressed as $f(n/2, n/2) = 1/h^2$, and $f(i, j) = 0$ for other grid points. Use the mesh and contour plots to visualize the solution for $n = 36$ $(mesh(x, y, u), contour(x, y, u, 30))$. Note that the solution is called the discrete Green function.