MA 584
Homework #3
Due October 25, 2020

1. For the implicit Euler's method applied to the heat equation $u_t = u_{xx}$, is it possible to choose Δt such that the discretization has order $O((\Delta t)^2 + h^4)$? What's the advantage of the implicit Euler's method?

2. Consider the diffusion and and advection equation

$$u_t + \epsilon u_x = (\beta u_x)_x + f(x,t), \quad \beta(x) \geq \beta_0 > 0. \tag{1}$$

   (a) Assuming $\epsilon = 1$, $f(x,t) = 0$, and β is a constant. Use the von Neumann analysis to derive the time step restriction for the following scheme

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} + \frac{U_{i+1}^k - U_{i-1}^k}{2h} = \beta\, \frac{U_{i-1}^k - 2U_i^k + U_{i+1}^k}{h^2}.$$

   (b) Implement the ***Crank-Nicolson*** or the ***MOL*** methods with the central/upwind discretization in space to solve the equation with a Dirchlet boundary condition at $x = a$ and a Neumann or Robin boundary condition at $x = b$.

   Use $u(x,t) = (\cos t)\, x^2 \sin(\pi x)$ to find the $f(x,t)$, $a < x < b$, $tfinal = 5.0$ to test and debug your code. Your discussion should include the grid refinement analysis, error and solution plots for $m = 80$.

   Hint: You can use Matlab code *ode15s* or *ode23s* to solve the semi-discrete ODE system.

   Assume that $\epsilon = 0$ and u is the temperature of a thin rod whose one end ($x = b$) has just been heated. The other end of the rod has a room temperature ($70^o F$ or $21^o C$). Solve the problem and find the history of the solution. Roughly how long the rod will reach the steady state solution? What is the exact solution of the steady state? **Hint:** Take the initial condition as $u(x,0) = T_0\, e^{-(x-b)^2/\gamma}$, where $T_0 = 100^o C$ and $\gamma = 2$ are two constants, $f(x,t) = 0$, and the the Neumann boundary condition $u_x(b,t) = 0$.

3. Show that the Crank-Nicholson scheme is unconditionally stable for $u_t = \beta u_{xx}$ with Dirichlet boundary condition, where $\beta > 0$ is a constant.

   Extra credit: Consider the stability of that the Crank-Nicholson scheme for $u_t = \beta u_{xx} + \alpha u_x$ when the u_x is discretized by the central scheme; and/or the up-winding scheme.

4. Write down the coefficient matrix of the finite difference method using the standard central five point stencil with the Red-Black for the Poisson equation defined on the rectangle $[a,\ b] \times [c,\ d]$. Take $m = n = 3$ and assume a Dirichlet boundary condition at $x = a$, $y = c$ and $y = d$, and a Neumann boundary condition $\frac{\partial u}{\partial n} = g(y)$ at $x = b$. Use the ghost point method to deal with the Neumann boundary condition.

5. (a) Download the instructor's Matlab code to solve Poisson equations $u_{xx}+u_{yy} = f(x,y)$ on rectangular domains for $(a):\ u(x,y) = e^x \sin y$, what is $f(x,y)$?; $(b):\ u(x,y) = \sin(k_1 x)\sin(k_2 y)$, what is $f(x,y)$?. Try $k_1 = 5$, $k_2 = 3$ and $k_1 = 100$, $k_2 = 3$. Carry out the grid refinement analysis to find the convergence order. Explain and analysis your results as much as you can.

   (b) Modify your code to solve $u_{xx} + u_{yy} + \sigma u = f(x,y)$ to solve the same problems above. Try $\sigma = -5$, $\sigma = 5$ and $\sigma = -500$, $\sigma = 500$. Explain and analysis your results as much as you can.