MA 584
Homework #2
Due September 28, 2020

Note: The first two problems are paper-work problems. The last two problems are both paper-work and coding problems.

1. Given
$$u''(x) - q(x)u(x) = f(x), \quad a < x < b,$$
$$u(a) = u_a, \qquad u'(b) = \beta,$$
where u_a and β are constants.
   (a) What conditions (sufficient) should we impose on $f(x)$, $q(x)$ so that the problem is well-posed?
   (b) If we use the ghost point method to discretize $u'(b) = \beta$, write down $A_h U = F$. Show that the local truncation error of the FD scheme is $O(h)$ at $x_n = b$. What is the global error of the FD method?
   (c) If we use the forward Euler to discretize $u'(b) = \beta$, write down $A_h U = F$. Show that the local truncation error of the FD scheme is $O(1)$ at $x_n = b$ except when $\beta = 0$ (O(?)). What is the global error of the method when $\beta = 0$ and $\beta \neq 0$?

2. Derive a finite difference method for
$$u''(x) - q(x)u(x) = f(x), \quad a < x < b, \tag{1}$$
$$\text{periodic BC, that is,} \quad u(a) = u(b), \quad u'(a) = u'(b), \cdots, \tag{2}$$
using the central difference scheme and a uniform grid. Write down the system of equations $A_h U = F$. How many unknowns are there without redundant? Is the coefficient matrix A_h tri-diagonal? **Hint:** Note that $U_0 = U_n$. Set unknowns as U_1, U_2, $\cdots$, U_n. If $q(x) = 0$, does the solution exist? Derive a compatibility condition for which the solution exists. If the solution exists, is it unique? How do we modify the finite difference method to make the solution unique?

3. Derive and implement a finite difference method to solve the Sturm-Liouville eigenvalue problem
$$(pu')' + qu = \lambda u, \qquad a < x < b. \tag{3}$$
$$u(a) = 0, \qquad u(b) = 0. \tag{4}$$
That is, find a pair $(u(x), \lambda)$ that satisfy the ODE and BCs and $u(x) \neq 0$. Particularly, check your code for $p(x) = 1$, $q(x) = 0$, $a = 0$, $b = \pi$. The analytic solution is $\lambda_n = -n^2$, $n = 1, 2, \cdots$, and corresponding $u(x) = \sin nx$. List all eigenvalues and plot several normalized eigenfunctions ($\|u_k\| = 1, k = 1, 5, 20, 78$ with the mesh size $N = 80$. **Hint:** in Matlab, use $[U, D] = eig(A)$ to find the eigenvalues and eigenvectors. You can use subplot(2,2,1), subplot(2,2,2),subplot(2,2,3),subplot(2,2,4), to put four plots together.

4. Consider the finite difference scheme for the 1D steady state *convection-diffusion* equation
$$\epsilon u'' - u' = -1, \quad 0 < x < 1 \tag{5}$$
$$u(0) = 1, \qquad u(1) = 3. \tag{6}$$

(a) Verify the exact solution is $u(x) = 1 + x + \left(\dfrac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}\right)$.

(b) Can we re-write the ODE as a self-adjoint form?

(c) Compare the following two finite difference methods for $\epsilon = 0.3$, 0.1, 0.05, and 0.0005,
(1): Central difference scheme:

$$\epsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - \frac{U_{i+1} - U_{i-1}}{2h} = -1. \tag{7}$$

(2): Central-upwind difference scheme:

$$\epsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - \frac{U_i - U_{i-1}}{h} = -1. \tag{8}$$

Do grid refinement analysis for each case to determine the order of accuracy. Plot the computed solution and the exact solution for $h = 0.1$, $h = 1/25$, and $h = 0.01$. You can use Matlab command *subplot* to put several graphs together.

(d) From you observation, give your opinion to see which method is better. (**Hint:** The answer may not be unique and depends on the solution and parameters.)

5. **Extra credit:** Consider

$$u''(x) = f(x), \qquad a < x < b, \qquad u(a) = u_a, \quad u(b) = u_b. \tag{9}$$

Derive, implement, and validate a fourth order finite difference scheme to solve the problem.

Hint: The second order centered finite difference operator is

$$\delta_{xx}^2 u = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}. \tag{10}$$

Show that

$$\begin{aligned} \delta_{xx}^2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + O(h^4) \\ &= \left(1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2}{\partial x^2} u + O(h^4), \end{aligned} \tag{11}$$

and substituting the operator relation

$$\frac{\partial^2}{\partial x^2} = \delta_{xx}^2 + O(h^2)$$

into the equation, we obtain

$$\begin{aligned} \delta_{xx}^2 u &= \left(1 + \frac{h^2}{12}\left(\delta_{xx}^2 + O(h^2)\right)\right)\frac{\partial^2}{\partial x^2} u + O(h^4) \\ &= \left(1 + \frac{h^2}{12}\delta_{xx}^2\right)\frac{\partial^2}{\partial x^2} u + O(h^4), \end{aligned}$$

from which we further have

$$\frac{\partial^2}{\partial x^2} = \left(1 + \frac{h^2}{12}\delta_{xx}^2\right)^{-1}\delta_{xx}^2 u + \left(1 + \frac{h^2}{12}\delta_{xx}^2\right)^{-1} O(h^4).$$