5

Finite Difference Methods for Hyperbolic PDEs

In this chapter, we discuss finite difference methods for hyperbolic PDEs (see page 6 for the definition of hyperbolic PDEs). Let us first list a few typical model problems involving hyperbolic PDEs.

- Advection equation (one-way wave equation):

$$\begin{gathered} u_t + au_x = 0\,, \quad 0 < x < 1\,, \\ u(x,0) = \eta(x)\,, \qquad \text{IC}\,, \\ u(0,t) = g_l(t) \quad \text{if} \quad a \geq 0\,, \quad \text{or} \quad u(1,t) = g_r(t) \quad \text{if} \quad a \leq 0\,. \end{gathered} \tag{5.1}$$

  Here g_l and g_r are prescribed boundary conditions from the left and right, respectively.
- Second-order linear wave equation:

$$\begin{gathered} u_{tt} = au_{xx}\,, \quad 0 < x < 1\,, \\ u(x,0) = \eta(x), \qquad \frac{\partial u}{\partial t}(x,0) = v(x), \qquad \text{IC}\,, \\ u(0,t) = g_l(t)\,, \qquad u(1,t) = g_r(t), \qquad \text{BC}. \end{gathered} \tag{5.2}$$

- Linear first-order hyperbolic system:

$$\mathbf{u}_t = A\mathbf{u}_x + \mathbf{f}(x,t)\,, \tag{5.3}$$

  where **u** and **f** are two vectors and A is a matrix. The system is called *hyperbolic* if A is diagonalizable, i.e., if there is a nonsingular matrix T such that $A = TDT^{-1}$, and all eigenvalues of A are real numbers.

- Nonlinear hyperbolic equation or system, notably conservation laws:

$$u_t + f(u)_x = 0\,, \quad \text{e.g., Burger's equation } u_x + \left(\frac{u^2}{2}\right)_x = 0\,; \tag{5.4}$$

$$\mathbf{u}_t + \mathbf{f}_x + \mathbf{g}_y = 0\,. \tag{5.5}$$

  For nonlinear hyperbolic PDE, shocks (a discontinuous solution) can develop even if the initial data is smooth.

5.1 Characteristics and Boundary Conditions

We know the exact solution for the one-way wave equation

$$u_t + au_x = 0\,, \quad -\infty < x < \infty\,,$$
$$u(x,0) = \eta(x)\,, \quad t > 0$$

is $u(x,t) = \eta(x - at)$. If the domain is finite, we can also find the exact solution. We can solve the model problem

$$u_t + au_x = 0\,, \quad 0 < x < 1\,,$$
$$u(x,0) = \eta(x)\,, \quad t > 0\,, \qquad u(0,t) = g_l(t) \quad \text{if } a > 0$$

by the *method of characteristics* since the solution is constant along the characteristics. For any point (x,t) we can readily trace the solution. In fact, for the characteristic

$$z(s) = u(x + ks, t + s) \tag{5.6}$$

along which the solution is a constant ($z'(s) \equiv 0$), on substituting into the PDE we get

$$z'(s) = u_t + ku_x = 0\,,$$

which is always true if $k = a$. The solution at $(x + ks, t + s)$ is the same as at (x,t), so we can solve the problem by tracing back until the line hits the boundary, i.e., $u(\bar{x},\bar{t}) = u(x + as, t + s) = u(x - at,\, 0)$ if $x - at \geq 0$, on tracing back to the initial condition. If $x - at < 0$, we can only trace back to $x = 0$ or $s = -\bar{x}/a$ and $t = \bar{x}/a$, and the solution is $u(\bar{x},\bar{t}) = u(0, t - \bar{x}/a) = g_l(t - \bar{x}/a)$. The solution for the case $a \geq 0$ can therefore be written as

$$u(x,t) = \begin{cases} \eta(x - at) & \text{if } x \geq at, \\ g_l\left(t - \dfrac{x}{a}\right) & \text{if } x < at\,. \end{cases} \tag{5.7}$$

Now we can see why we have to prescribe a boundary condition at $x = 0$, but we cannot have any boundary condition at $x = 1$. It is important to have correct boundary conditions for hyperbolic problems!

The one-way wave equation is often used as a benchmark problem for different numerical methods for hyperbolic problems.

5.2 Finite Difference Schemes

Simple numerical methods for hyperbolic problems include:

- Lax–Friedrichs method;
- Upwind scheme;
- Leap-frog method (note it does not work for the heat equation but works for linear hyperbolic equations);
- Box scheme;
- Lax–Wendroff method;
- Crank–Nicolson scheme (not recommended for hyperbolic problems, since there are no severe time step size constraints); and
- Beam–Warming method (one-sided second-order upwind scheme if the solution is smooth).

There are also some high-order methods in the literature. For linear hyperbolic problems, if the initial data in smooth (no discontinuities), it is recommended to use second-order accurate methods such as the Lax–Wendroff method. However, care has to be taken if the initial data has finite discontinuities, called shocks, as second- or high-order methods often lead to oscillations near the discontinuities (Gibbs phenomena). Some of the methods are the bases for numerical methods for conservation law, a special conservative nonlinear hyperbolic system, for which shocks may develop in finite time even if the initial data is smooth. Also for hyperbolic differential equations, usually there is no strict time step constraint as for the parabolic problems. Often explicit methods are preferred.

5.2.1 Lax–Friedrichs Method

Consider the one-way wave equation $u_t + au_x = 0$, and the simple finite difference scheme

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{a}{2h}\left(U_{j+1}^k - U_{j-1}^k\right) = 0\,,$$

$$\text{or} \qquad U_j^{k+1} = U_j^k - \mu\left(U_{j+1}^k - U_{j-1}^k\right),$$

where $\mu = a\Delta t/(2h)$. The scheme has $O(\Delta t + h^2)$ local truncation error, but the method is unconditionally unstable from the von Neumann stability analysis. The growth factor for the FW-CT finite difference scheme is

$$\begin{aligned} g(\theta) &= 1 - \mu \left(e^{ih\xi} - e^{-ih\xi} \right) \\ &= 1 - \mu\, 2\, i\, \sin(h\xi), \end{aligned}$$

where $\theta = h\xi$, so

$$|g(\theta)|^2 = 1 + 4\mu^2 \sin^2(h\xi) \geq 1\,.$$

In the Lax–Friedrichs scheme, we average U_j^k using U_{j-1}^k and U_{j+1}^k to get

$$U_j^{k+1} = \frac{1}{2}\left(U_{j-1}^k + U_{j+1}^k \right) - \mu \left(U_{j+1}^k - U_{j-1}^k \right).$$

The local truncation error is $O(\Delta t + h)$ if $\Delta t \simeq h$. The growth factor is

$$\begin{aligned} g(\theta) &= \frac{1}{2}\left(e^{ih\xi} + e^{-ih\xi} \right) + \mu \left(e^{ih\xi} - e^{-ih\xi} \right) \\ &= \cos(h\xi) - 2\mu \sin(h\xi) i \end{aligned}$$

so

$$\begin{aligned} |g(\theta)|^2 &= \cos^2(h\xi) + 4\mu^2 \sin^2(h\xi) \\ &= 1 - \sin^2(h\xi) + 4\mu^2 \sin^2(h\xi) \\ &= 1 - (1 - 4\mu^2) \sin^2(h\xi)\,, \end{aligned}$$

and we conclude that $|g(\theta)| \leq 1$ if $1 - 4\mu^2 \geq 0$ or $1 - (a\Delta t/h)^2 \geq 0$, which implies that $\Delta t \leq h/|a|$. This is the CFL (Courant–Friedrichs–Lewy) condition.

For the Lax–Friedrichs scheme, we need a NBC at $x = 1$, as explained later. The Lax–Friedrichs scheme is the basis for several other popular schemes. A MATLAB code called *lax_fred.m* can be found in the MATLAB programming collections that accompany the book.

5.2.2 The Upwind Scheme

The upwind scheme for $u_t + au_x = 0$ is

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} = \begin{cases} -\dfrac{a}{h}\left(U_j^k - U_{j-1}^k \right) & \text{if } a \geq 0\,, \\ -\dfrac{a}{h}\left(U_{j+1}^k - U_j^k \right) & \text{if } a < 0\,, \end{cases} \tag{5.8}$$

which is first-order accurate in time and in space. To find the CFL constraint, we conduct the von Neumann stability analysis. The growth factor for the case when $a \geq 0$ is

$$\begin{aligned} g(\theta) &= 1 - \mu\left(1 - e^{-ih\xi}\right) \\ &= 1 - \mu(1 - \cos(h\xi)) - i\mu\sin(h\xi) \end{aligned}$$

with magnitude

$$\begin{aligned} |g(\theta)|^2 &= (1 - \mu + \mu\cos(h\xi))^2 + \mu^2\sin^2(h\xi) \\ &= (1-\mu)^2 + 2(1-\mu)\mu\cos(h\xi) + \mu^2 \\ &= 1 - 2(1-\mu)\mu(1-\cos(h\xi)), \end{aligned}$$

so if $1 - \mu \geq 0$ (i.e., $\mu \leq 1$) or $\Delta t \leq h/a$ we have $|g(\theta)| \leq 1$.

Note that no NBC is needed for the upwind scheme, and there is no severe time step restriction, since $\Delta t \leq h/a$. If $a = a(x,t)$ is a variable function that does not change the sign, then the CFL condition is

$$0 < \Delta t \leq \frac{h}{\max|a(x,t)|}.$$

However, the upwind scheme is first-order in time and in space, and there are some high-order schemes.

A MATLAB code called upwind.m can be found in the MATLAB programming collections accompanying this book. The main structure of the code is listed below:

```
a = 0;  b=1; tfinal = 0.5  % Input the domain and final time.
m = 20; h = (b-a)/m; k = h; mu = k/h;  % Set mesh and time step

n = fix(tfinal/k);  % Find the number of time steps
y1 = zeros(m+1,1); y2=y1; x=y1;    % Initialization

figure(1);  hold        % Open a plot window for solutions at
                          different time.
axis([-0.1 1.1 -0.1 1.1]);

for i=1:m+1,            % Initialization.
  x(i) = a + (i-1)*h;
  y1(i) = uexact(t,x(i));  y2(i) = 0;
end

t = 0;     % Begin time marching.
for j=1:n,
  y1(1)=bc(t); y2(1)=bc(t+k);
  for i=2:m+1
    y2(i) = y1(i) - mu*(y1(i)-y1(i-1) );
  end
```

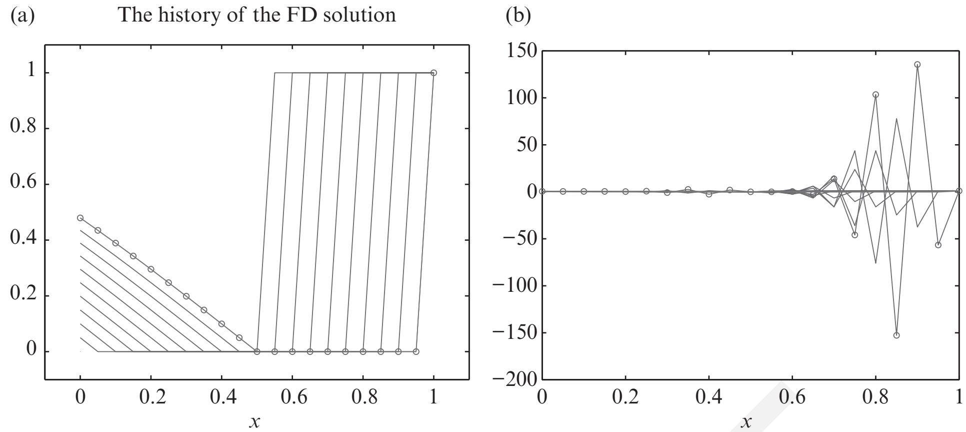

Fig. 5.1. Plot of the initial and consecutive approximation of the upwinding method for an advection equation. (a) The time step is $\Delta t = h$ and the scheme is stable. (b) The time step is $\Delta t = 1.5h$ and the scheme is unstable which leads to a blowing-up quantity.

```
  t = t + k;  y1 = y2;      % Overwrite old solutions
  plot(x,y2); pause(0.5)    % Plot the current solution.
end
```

In Figure 5.1, we show the initial data and several consecutive finite difference approximations of the upwinding scheme applied to the advection equation $u_t + u_x = 0$ in the domain $0 < x < 1$. The initial condition is

$$u(x,0) = u_0(x) = \begin{cases} 0 & \text{if } 0 < x < 1/2, \\ 1 & \text{if } 1/2 \le x < 1\,. \end{cases}$$

The boundary condition is $u(0,t) = \sin t$. The analytic solution is

$$u(x,t) = \begin{cases} u_0(x-t) & \text{if } 0 < t < x < 1, \\ \sin(t-x) & \text{if } 0 < x < t < 1\,. \end{cases}$$

If we take $\Delta t \le h$, the scheme works well and we obtained the exact solution for this example (see Figure 5.1a). However, if we take $\Delta t > h$, say $\Delta t = 1.5h$ as in the plot of Figure 5.1b, the solution blows-up quickly since the scheme is unstable. Once again, it shows the importance of the time step constraint.

5.2.3 The Leap-Frog Scheme

The leap-frog scheme for $u_t + au_x = 0$ is

$$\begin{aligned} \frac{U_j^{k+1} - U_j^{k-1}}{2\Delta t} + \frac{a}{2h}\left(U_{j+1}^k - U_{j-1}^k\right) = 0\,, \\ \text{or} \qquad U_j^{k+1} = U_j^{k-1} - \mu\left(U_{j+1}^k - U_{j-1}^k\right), \end{aligned} \tag{5.9}$$

where $\mu = a\Delta t/(2h)$. The discretization is second-order in time and in space. It requires a NBC at one end and needs U_j^1 to get started. We know that the leap-frog scheme is unconditionally unstable for the heat equation. Let us consider the stability for the advection equation through the von Neumann analysis. Substituting

$$U_j^k = e^{ij\xi}, \quad U_j^{k+1} = g(\xi)e^{ij\xi}, \quad U_j^{k-1} = \frac{1}{g(\xi)}\,e^{ij\xi}$$

into the leap-frog scheme, we get

$$\begin{aligned} g^2 + \mu(e^{ih\xi} - e^{-ih\xi})g - 1 = 0\,, \\ \text{or} \qquad g^2 + 2\mu i \sin(h\xi)\, g - 1 = 0\,, \end{aligned}$$

with solution

$$g_\pm = -i\mu\sin(h\xi) \pm \sqrt{1 - \mu^2\sin^2(h\xi)}\,. \tag{5.10}$$

We distinguish three different cases.

1. If $|\mu| > 1$, then there are ξ such that at least one of $|g_-| > 1$ or $|g_+| > 1$ holds, so the scheme is unstable!
2. If $|\mu| < 1$, then $1 - \mu^2\sin^2(h\xi) \geq 0$ such that

   $$|g_\pm|^2 = \mu^2\sin^2(h\xi) + 1 - \mu^2\sin^2(h\xi) = 1\,.$$

   However, since it is a two-stage method, we have to be careful about the stability. From linear finite difference theory, we know the general solution is

   $$\begin{aligned} U^k &= C_1 g_-^k + C_2 g_+^k \\ |U^k| &\leq \max\{C_1, C_2\}\left(|g_-^k| + |g_+^k|\right) \\ &\leq 2\max\{C_1, C_2\}\,, \end{aligned}$$

   so the scheme is neutral stable according to the definition $\|\mathbf{U}^k\| \leq C_T \sum_{j=0}^J \|\mathbf{U}^j\|$.

3. If $|\mu| = 1$, we still have $|g_\pm| = 1$, but we can find ξ such that $\mu \sin(h\xi) = 1$ and $g_+ = g_- = -i$, i.e., $-i$ is a double root of the characteristic polynomial. The solution of the finite difference equation therefore has the form

$$U_j^k = C_1(-i)^k + C_2 k(-i)^k\,,$$

where the possibly complex numbers C_1 and C_2 are determined from the initial conditions. Thus there are solutions such that $\|\mathbf{U}^k\| \simeq k$ which are unstable (slow growing).

In conclusion, the leap-frog scheme is stable if $\Delta t < \frac{h}{|a|}$. Note that we can use the upwind or other scheme (even unstable ones) to initialize the leap-frog scheme to get U_j^1. We call a numerical scheme (such as the Lax–Friedrichs and upwind schemes) dissipative if $|g(\xi)| < 1$, and otherwise (such as the leap-frog scheme) it is nondissipative.

5.3 The Modified PDE and Numerical Diffusion/Dispersion

A modified PDE is the PDE that a finite difference equation satisfies exactly at grid points. Consider the upwind method for the advection equation $u_t + au_x = 0$ in the case $a > 0$,

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{a}{h}\left(U_j^k - U_{j-1}^k\right) = 0\,.$$

The derivation of a modified PDE is similar to computing the local truncation error, only now we insert $v(x,t)$ into the finite difference equation to derive a PDE that $v(x,t)$ satisfies exactly, thus

$$\frac{v(x.t+\Delta t) - v(x,t)}{\Delta t} + \frac{a}{h}\left(v(x,t) - v(x-h,t)\right) = 0\,.$$

Expanding the terms in Taylor series about (x,t) and simplifying yields

$$v_t + \frac{1}{2}\Delta t\, v_{tt} + \cdots + a\left(v_x - \frac{1}{2}hv_{xx} + \frac{1}{6}h^2 v_{xxx} + \cdots\right) = 0\,,$$

which can be rewritten as

$$v_t + av_x = \frac{1}{2}(ahv_{xx} - \Delta t v_{tt}) - \frac{1}{6}\left(ah^2 v_{xxx} + (\Delta t)^2 v_{ttt}\right) + \cdots\,,$$

which is the PDE that v satisfies. Consequently,

$$\begin{aligned} v_{tt} &= -av_{xt} + \frac{1}{2}(ahv_{xxt} - \Delta t v_{ttt}) \\ &= -av_{xt} + O(\Delta t, h) \\ &= -a\frac{\partial}{\partial x}\left(-av_x + O(\Delta t, h)\right), \end{aligned}$$

so the leading modified PDE is

$$v_t + av_x = \frac{1}{2}ah\left(1 - \frac{a\Delta t}{h}\right)v_{xx}\,. \tag{5.11}$$

This is a advection–diffusion equation. The grid values U_j^n can be viewed as giving a second-order accurate approximation to the true solution of this equation, whereas they only give a first-order accurate approximation to the true solution of the original problem. From the modified equation, we can conclude that:

- the computed solution smooths out discontinuities because of the diffusion term (the second-order derivative term is called numerical dissipation, or numerical viscosity);
- if a is a constant and $\Delta t = h/a$, then $1 - a\Delta t/h = 0$ (we have second-order accuracy);
- we can add the correction term to offset the leading error term to render a higher-order accurate method, but the stability needs to be checked, For instance, we can modify the upwind scheme to get

  $$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a\frac{U_j^k - U_{j-1}^k}{h} = \frac{1}{2}ah\left(1 - \frac{a\Delta t}{h}\right)\frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2}\,,$$

  which is second-order accurate if $\Delta t \simeq h$;
- from the modified equation, we can see why some schemes are unstable, e.g., the leading term of the modified PDE for the unstable scheme

  $$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a\frac{U_{j+1}^k - U_{j-1}^k}{2h} = 0 \tag{5.12}$$

  is

  $$v_t + av_x = -\frac{a^2\Delta t}{2}v_{xx}\,, \tag{5.13}$$

  where the highest derivative is similar to the backward heat equation that is dynamically unstable!

5.4 The Lax–Wendroff Scheme and Other FD methods

To derive the Lax–Wendroff scheme, we notice that

$$\begin{aligned}\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} &= u_t + \frac{\Delta t}{2}u_{tt} + O((\Delta t)^2)\\ &= u_t + \frac{1}{2}a^2(\Delta t)u_{xx} + O((\Delta t)^2)\,.\end{aligned}$$

We can add the numerical viscosity $\frac{1}{2}a^2\Delta t\, u_{xx}$ term to improve the accuracy in time, to get the Lax–Wendroff scheme

$$\frac{U_j^{k+1}-U_j^k}{\Delta t} + a\frac{U_{j+1}^k - U_{j-1}^k}{2h} = \frac{1}{2}\frac{a^2\Delta t}{h^2}\left(U_{j-1}^k - 2U_j^k + U_{j+1}^k\right), \quad (5.14)$$

which is second-order accurate both in time and space. To show this, we investigate the corresponding local truncation error

$$\begin{aligned}T(x,t) &= \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} - \frac{a\,(u(x+h,t)-u(x-h,t))}{2h}\\ &\quad - \frac{a^2\Delta t\,(u(x-h,t)-2u(x,t)+u(x+h,t))}{2h^2}\\ &= u_t + \frac{\Delta t}{2}u_{tt} - au_x - \frac{a^2\Delta t}{2}u_{xx} + O((\Delta t)^2+h^2)\\ &= O((\Delta t)^2+h^2)\,,\end{aligned}$$

since $u_t = -au_x$ and $u_{tt} = -au_{xt} = -a\frac{\partial}{\partial x}u_t = a^2u_{xx}$.

To get the CFL condition for the Lax–Wendroff scheme, we carry out the von Neumann stability analysis. The growth factor of the Lax–Wendroff scheme is

$$\begin{aligned}g(\theta) &= 1 - \frac{\mu}{2}\left(e^{ih\xi} - e^{-ih\xi}\right) + \frac{\mu^2}{2}\left(e^{-ih\xi} - 2 + e^{ih\xi}\right)\\ &= 1 - \mu i\sin\theta - 2\mu^2\sin^2(\theta/2),\end{aligned}$$

where again $\theta = h\xi$, so

$$
\begin{aligned}
|g(\theta)|^2 &= \left(1 - 2\mu^2 \sin^2 \frac{\theta}{2}\right)^2 + \mu^2 \sin^2 \theta \\
&= 1 - 4\mu^2 \sin^2 \frac{\theta}{2} + 4\mu^4 \sin^4 \frac{\theta}{2} + 4\mu^2 \sin^2 \frac{\theta}{2} \left(1 - \sin^2 \frac{\theta}{2}\right) \\
&= 1 - 4\mu^2 \left(1 - \mu^2\right) \sin^4 \frac{\theta}{2} \\
&\leq 1 - 4\mu^2 \left(1 - \mu^2\right).
\end{aligned}
$$

We conclude $|g(\theta)| \leq 1$ if $\mu \leq 1$, i.e., $\Delta t \leq h/|a|$. If $\Delta t > h/|a|$, there are ξ such that $|g(\theta)| > 1$ so the scheme is unstable.

The leading modified PDE for the Lax–Wendroff method is

$$
v_t + av_x = -\frac{1}{6}ah^2 \left(1 - \left(\frac{a\Delta t}{h}\right)^2\right) v_{xxx} \tag{5.15}
$$

which is a dispersive equation. The group velocity for the wave number ξ is

$$
c_g = a - \frac{1}{2}ah^2 \left(1 - \left(\frac{a\Delta t}{h}\right)^2\right) \xi^2 , \tag{5.16}
$$

which is less than a for all wave numbers. Consequently, the numerical result can be expected to develop a train of oscillations behind the peak, with high wave numbers lagging farther behind the correct location (*cf.* Strikwerda, 1989) for more details. If we retain one more term in the modified equation for the Lax–Wendroff scheme, we get

$$
v_t + av_x = \frac{1}{6}ah^2 \left(\left(\frac{a\Delta t}{h}\right)^2 - 1\right) v_{xxx} - \epsilon v_{xxxx} , \tag{5.17}
$$

where the ϵ in the fourth-order dissipative term is $O(h^3)$ and positive when the stability bound holds. This high-order dissipation causes the highest wave number to be damped, so that the oscillations are limited.

5.4.1 The Beam–Warming Method

The Beam–Warming method is a one-sided finite difference scheme for the modified equation

$$
v_t + av_x = \frac{a^2 \Delta t}{2} v_{xx} .
$$

Recall the one-sided finite difference formulas, *cf*. page 21

$$u'(x) = \frac{3u(x) - 4u(x-h) + u(x-2h)}{2h} + O(h^2)\,,$$
$$u''(x) = \frac{u(x) - 2u(x-h) + u(x-2h)}{h^2} + O(h)\,.$$

The Beam–Warming method for $u_t + au_x = 0$ for $a > 0$ is

$$U_j^{k+1} = U_j^k - \frac{a\Delta t}{2h}\left(3U_j^k - 4U_{j-1}^k + U_{j-2}^k\right) + \frac{(a\Delta t)^2}{2h^2}\left(U_j^k - 2U_{j-1}^k + U_{j-2}^k\right), \tag{5.18}$$

which is second-order accurate in time and space if $\Delta t \simeq h$. The CFL constraint is

$$0 < \Delta t \le \frac{2h}{|a|}\,. \tag{5.19}$$

For this method, we do not require a NBC at $x = 1$, but we need a scheme to compute the solution U_1^j. The leading terms of the modified PDE for the Beam–Warming method are

$$v_t + av_x = \frac{1}{6}ah^2\left(\left(\frac{a\Delta t}{h}\right)^2 - 1\right)v_{xxx}\,. \tag{5.20}$$

In this case, the group velocity is greater than a for all wave numbers when $0 \le a\Delta t/h \le 1$, so initial oscillations would move ahead of the main hump. On the other hand, if $1 \le a\Delta t/h \le 2$ the group velocity is less than a, so the oscillations fall behind.

5.4.2 The Crank–Nicolson Scheme

The Crank–Nicolson scheme for the advection equation $u_t + au_x = f$ is

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a\frac{U_{j+1}^k - U_{j-1}^k + U_{j+1}^{k+1} - U_{j-1}^{k+1}}{4h} = f_j^{k+\frac{1}{2}}\,, \tag{5.21}$$

which is second-order accurate in time and in space, and unconditionally stable. A NBC is needed at $x = 1$. This method is effective for the 1D problem, since it is easy to solve the resulting tridiagonal system of equations. For higher-dimensional problems, the method is not recommended in general as for hyperbolic equations the time step constraint $\Delta t \simeq h$ is not a major concern.

5.4.3 The Method of Lines

Different method of lines (MOL) methods can be used, depending on how the spatial derivative term is discretized. For the advection equation $u_t + au_x = 0$, if we use

$$\frac{\partial U_i}{\partial t} + a\frac{U_{i+1} - U_{i-1}}{2h} = 0 \tag{5.22}$$

the ODE solver is likely to be implicit, since the leap-frog method is unstable!

5.5 Numerical Boundary Conditions

We need a numerical boundary condition (NBC) at one end for the one-way wave equation when we use any of the Lax–Friedrichs, Lax–Wendroff, or leap-frog schemes. There are several possible approaches.

- Extrapolation. One simplest first-order approximation is

  $$U_M^{k+1} = U_{M-1}^{k+1}.$$

  To get a second-order approximation, recall the Lagrange interpolation formula

  $$f(x) \simeq f(x_1)\frac{x - x_2}{x_1 - x_2} + f(x_2)\frac{x - x_1}{x_2 - x_1}\,.$$

  We can use the same time level for the interpolation to get

  $$U_M^{k+1} = U_{M-2}^{k+1}\frac{x_M - x_{M-1}}{x_{M-1} - x_M} + U_{M-1}^{k+1}\frac{x_M - x_{M-2}}{x_{M-2} - x_{M-1}}\,.$$

  If a uniform grid is used with spatial step size h, this formula becomes

  $$U_M^{k+1} = -U_{M-2}^{k+1} + 2U_{M-1}^{k+1}\,.$$

- Quasi-characteristics. If we use previous time levels for the interpolation, we get

  $$U_M^{k+1} = U_{M-1}^{k}, \quad \text{first order,}$$

  $$U_M^{k+1} = U_{M-2}^{k}\frac{x_M - x_{M-1}}{x_{M-1} - x_M} + U_{M-1}^{k}\frac{x_M - x_{M-2}}{x_{M-2} - x_{M-1}}\,, \quad \text{second order.}$$

- We may use schemes that do not need NBC at or near the boundary, e.g., the upwind scheme or the Beam–Warming method to provide the boundary conditions.

The accuracy and stability of numerical schemes usually depend upon the NBC used. Usually, the main scheme and the scheme for NBC should both be stable.

5.6 Finite Difference Methods for Second-Order Linear Hyperbolic PDEs

In reality, a 1D sound wave propagates in two directions and can be modeled by the wave equation

$$u_{tt} = a^2 u_{xx}, \tag{5.23}$$

where $a > 0$ is the wave speed. We can find the general solution by changing variables as follows,

$$\begin{cases} \xi = x - at \\ \eta = x + at \end{cases} \quad \text{or} \quad \begin{cases} x = \dfrac{\xi + \eta}{2} \\ t = \dfrac{\eta - \xi}{2a} \end{cases} \tag{5.24}$$

and using the chain-rule, we have

$$\begin{aligned} u_t &= -au_\xi + au_\eta \,, \\ u_{tt} &= a^2 u_{\xi\xi} - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta} \,, \\ u_x &= u_\xi + u_\eta \,, \\ u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \,. \end{aligned}$$

Substituting these relations into the wave equation, we get

$$u_{\xi\xi} a^2 - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta} = a^2 \left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \right),$$

which simplifies to

$$4a^2 u_{\xi\eta} = 0 \,,$$

yielding the solution

$$\begin{aligned} &u_\xi = \tilde{F}(\xi), \quad \Longrightarrow \quad u(x,t) = F(\xi) + G(\eta) \,, \\ &u(x,t) = F(x - at) + G(x + at) \,, \end{aligned}$$

where $F(\xi)$ and $G(\eta)$ are two differential functions of one variables. The two functions are determined by initial and boundary conditions.

With the general solution above, we can get the analytic solution to the Cauchy problem below:

$$\begin{aligned} &u_{tt} = a^2 u_{xx} \,, \qquad -\infty < x < \infty \,, \\ &u(x,0) = u_0(x) \,, \quad u_t(x,0) = g(x) \,, \end{aligned}$$

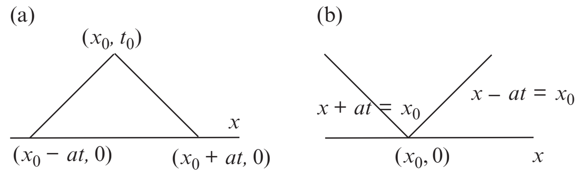

Fig. 5.2. A diagram of the domain of dependence (a) and influence (b).

as

$$u(x,t) = \frac{1}{2}\left(u_0(x-at) + u_0(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds. \qquad (5.25)$$

The solution is called the D'Alembert's formula. In particular, if $u_t(x,0) = 0$, then the solution is

$$u(x,t) = \frac{1}{2}\Big(u(x-at,0) + u(x+at,0)\Big),$$

demonstrating that a signal (wave) propagates along each characteristic $x - at$ and $x + at$ with speed a at half of its original strength. The solution $u(x,t)$ at a point (x_0, t_0) depends on the initial conditions only in the interval of $(x_0 - at_0, x_0 + at_0)$. The initial values between $(x_0 - at_0, x_0 + at_0)$ not only determines the solution value of $u(x,t)$ at (x_0, t_0) but also all the values of $u(x,t)$ in the triangle formed by the three vertices (x_0, t_0), $(x_0 - at_0,\ 0)$, and $(x_0 + at_0,\ 0)$. This domain is called the domain of dependence (see Figure 5.2a).

Also we see that given any point $(x_0,\ 0)$, any solution value $u(x,t)$, $t > 0$, in the cone formed by the characteristic lines $x + at = x_0$ and $x - at = x_0$ depends on the initial values at $(x_0,\ 0)$. The domain formed by the cone is called the domain of influence (see Figure 5.2b).

5.6.1 A FD Method (CT–CT) for Second-Order Wave Equations

Now we discuss how to solve the boundary value problems for which the analytic solution is difficult to get.

$$u_{tt} = a^2 u_{xx}\,, \qquad 0 < x < 1\,,$$

$$\text{IC:} \quad u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,,$$

$$\text{BC:} \quad u(0,t) = g_1(t)\,, \quad u(1,t) = g_2(t)\,.$$

We can use the central finite difference discretization both in time and space to get

$$\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{(\Delta t)^2} = a^2 \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2}\,, \tag{5.26}$$

which is second-order accurate both in time and space $((\Delta t)^2 + h^2)$. The CFL constraint for this method is $\Delta t \le \frac{h}{|a|}$, as verified through the following discussion.

The scheme above cannot be used to obtain the values of U_j^1 since $U_j^{-1} \sim u(x_j, -\Delta t)$ is not explicitly defined. There are several ways to jump-start the process. We list two commonly used ones below.

- Apply the forward Euler's method to the boundary condition $u_t(x,0) = u_1(x)$ to get $U_j^1 = U_j^0 + \Delta t\, u_1(x_j)$. The finite difference solution in a finite time $t = T$ will still be second-order accurate.
- Apply the ghost point method using $U_j^{-1} = U_j^1 - 2\Delta t\, u_1(x_j)$ to get

$$U_j^1 = U_j^0 + \Delta t\, u_1(x_j) + \left(\frac{a^2}{h}\right)^2 \left(U_{j-1}^0 - 2U_j^0 + U_{j+1}^0\right). \tag{5.27}$$

5.6.1.1 The Stability Analysis

The von Neumann analysis gives

$$\frac{g - 2 + 1/g}{(\Delta t)^2} = a^2 \frac{e^{-ih\xi} - 2 + e^{ih\xi}}{h^2}\,.$$

When $\mu = |a|\Delta t/h$, using $1 - \cos(h\xi) = 2\sin^2(h\xi/2)$, this equation becomes

$$g^2 - 2g + 1 = \left(-4\mu^2 \sin^2\theta\right) g,$$

or

$$g^2 - \left(2 - 4\mu^2\sin^2\theta\right) g + 1 = 0,$$

where $\theta = h\xi/2$, with solution

$$g = 1 - 2\mu^2\sin^2\theta \pm \sqrt{(1 - 2\mu^2\sin^2\theta)^2 - 1}\,.$$

Note that $1 - 2\mu^2\sin^2\theta \le 1$. If we also have $1 - 2\mu^2\sin^2\theta < -1$, then one of roots is

$$g_1 = 1 - 2\mu^2\sin^2\theta - \sqrt{(1 - 2\mu^2\sin^2\theta)^2 - 1} < -1$$

so $|g_1| > 1$ for some θ, such that the scheme is unstable.

To have a stable scheme, we require $1 - 2\mu^2 \sin^2\theta \geq -1$, or $\mu^2 \sin^2\theta \leq 1$, which can be guaranteed if $\mu^2 \leq 1$ or $\Delta t \leq h/|a|$. This is the CFL condition expected. Under this CFL constraint,

$$|g|^2 = \left(1 - 2\mu^2 \sin^2\theta\right)^2 + \left(1 - \left(1 - 2\mu^2 \sin^2\theta\right)^2\right) = 1$$

since the second part in the expression of g is imaginary, so the scheme is neutrally stable.

A finite difference scheme for a second-order PDE (in time) $P_{\Delta t,h} v_j^k = 0$ is stable in a stability region Λ if there is an integer J such that for any positive time T there is a constant C_T independent of Δt and h, such that

$$\|\mathbf{v}^n\|_h \leq \sqrt{1+n^2}\, C_T \sum_{j=0}^{J} \|\mathbf{v}^j\|_h \tag{5.28}$$

for any n that satisfies $0 \leq n\Delta t \leq T$ with $(\Delta t, h) \in \Lambda$. The definition allows linear growth in time. Once again, a finite difference scheme converges if it is consistent and stable.

5.6.2 Transforming the Second-Order Wave Equation to a First-Order System

Although we can solve the second-order wave equation directly, in this section, let us discuss how to change this equation into a first-order system. The first-order linear hyperbolic system of interest has the form

$$\mathbf{u}_t = (A\mathbf{u})_x = A\mathbf{u}_x\,,$$

which is a special case of 1D conservation laws

$$\mathbf{u}_t + (\mathbf{f}(\mathbf{u}))_x = 0.$$

To transfer the second-order wave equation to a first-order system, let us consider

$$\begin{cases} p = u_t \\ q = u_x\,, \end{cases} \qquad u_{tt} = p_t, \quad q_x = u_{xx}\,,$$

then we have

$$\begin{cases} p_t = u_{tt} = u_{xx} = q_x \\ q_t = u_{xt} = (u_t)_x = p_x \end{cases}$$

or in matrix-vector form

$$\begin{bmatrix} p \\ q \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_x , \tag{5.29}$$

and the eigenvalues of A are -1 and 1.

5.6.2.1 Initial and Boundary Conditions for the System

From the given boundary conditions for $u(x,t)$, we get

$$u(0,t) = g_1(t) , \qquad u_t(0,t) = g_1'(t) = p(0,t) ,$$
$$u(1,t) = g_2(t) , \qquad u_t(0,t) = g_1'(t) = p(1,t) ,$$

and there is no boundary condition for $q(x,t)$. The initial conditions are

$$p(x,0) = u_t(x,0) = u_1(x), \quad \text{known},$$
$$q(x,0) = u_x(x,0) = \frac{\partial}{\partial x} u(x,0) = u_0'(x), \quad \text{known}.$$

To solve the first-order system $\mathbf{u}_t = A\mathbf{u}_x$ numerically, we usually diagonalize the system (corresponding to characteristic directions) and then determine the boundary conditions, and apply an appropriate numerical method (e.g., the upwind method). Thus we write $A = T^{-1}DT$, where $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the matrix containing the eigenvalues of A on the diagonal and T is a nonsingular matrix. From the following

$$\mathbf{u}_t = A\mathbf{u}_x , \qquad T\mathbf{u}_t = TAT^{-1}T\mathbf{u}_x , \qquad (T\mathbf{u})_t = D\,(T\mathbf{u})_x ,$$

and writing $\tilde{\mathbf{u}} = T\mathbf{u}$, we get the new first-order system

$$\tilde{\mathbf{u}}_t = D\tilde{\mathbf{u}}_x$$

or $(\tilde{u}_i)_t = \lambda_i(\tilde{u}_i)_x$, $i = 1, 2, \ldots, n$, a simple system of equations that we can solve one by one. We also know at which end we should have a boundary condition, depending on the sign of λ_i.

For the second-order wave equation, let us recall that the eigenvalues are 1 and -1. The unit eigenvector (such that $\|x\|_2 = 1$) corresponding to the eigenvalue 1, found by solving $Ax = x$, is $x = [1,\ 1]^T/\sqrt{2}$. Similarly, the unit eigenvector corresponding to the eigenvalue -1 is $x = [-1,\ 1]^T/\sqrt{2}$, so

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} .$$

The transformed result is thus

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_t = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_x$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}_x ,$$

or in equivalent component form

$$\left(\frac{1}{\sqrt{2}}p - \frac{1}{\sqrt{2}}q\right)_t = -\left(\frac{1}{\sqrt{2}}p - \frac{1}{\sqrt{2}}q\right)_x$$
$$\left(\frac{1}{\sqrt{2}}p + \frac{1}{\sqrt{2}}q\right)_t = \left(\frac{1}{\sqrt{2}}p + \frac{1}{\sqrt{2}}q\right)_x .$$

By setting

$$\begin{cases} y_1 = \frac{1}{\sqrt{2}}p - \frac{1}{\sqrt{2}}q\,, \\ y_2 = \frac{1}{\sqrt{2}}p + \frac{1}{\sqrt{2}}q\,, \end{cases}$$

we get

$$\begin{cases} \frac{\partial}{\partial t}y_1 = -\frac{\partial}{\partial x}y_1\,, \\ \frac{\partial}{\partial t}y_2 = \frac{\partial}{\partial x}y_2\,, \end{cases}$$

i.e, two separate one-way wave equations, for which we can use various numerical methods.

We already know the initial conditions, but need to determine a boundary condition for y_1 at $x = 0$ and a boundary condition for y_2 at $x = 1$. Note that

$$y_1(0,t) = \frac{1}{\sqrt{2}}p(0,t) - \frac{1}{\sqrt{2}}q(0,t),$$

$$y_2(0,t) = \frac{1}{\sqrt{2}}p(0,t) + \frac{1}{\sqrt{2}}q(0,t),$$

and $q(0,t)$ is unknown. We do know that, however,

$$y_1(0,t) + y_2(0,t) = \frac{2}{\sqrt{2}} p(0,t)\,,$$

and can use the following steps to determine the boundary condition at $x = 0$:

1. update $(y_1)_0^{k+1}$ first, for which we do not need a boundary condition; and
2. use $(y_2)_0^{k+1} = \frac{2}{\sqrt{2}} p_0^{k+1} - (y_1)_0^{k+1}$ to get the boundary condition for y_2 at $x = 0$.

Similar steps likewise can be applied to the boundary condition at $x = 1$ as well.

5.7 Some Commonly Used FD Methods for Linear System of Hyperbolic PDEs

We now list some commonly used finite difference methods for solving a linear hyperbolic system of PDEs

$$\mathbf{u}_t + A\mathbf{u}_x = 0\,, \qquad \mathbf{u}(x,0) = \mathbf{u}_0(x)\,, \tag{5.30}$$

where A is a matrix with real eigenvalues.

- The Lax–Friedrichs scheme
$$\mathbf{U}_j^{k+1} = \frac{1}{2}\left(\mathbf{U}_{j+1}^k + \mathbf{U}_{j-1}^k\right) - \frac{\Delta t}{2h} A \left(\mathbf{U}_{j+1}^k - \mathbf{U}_{j-1}^k\right). \tag{5.31}$$
- The leap-frog scheme
$$\mathbf{U}_j^{k+1} = \mathbf{U}_j^{k-1} - \frac{\Delta t}{2h} A \left(\mathbf{U}_{j+1}^k - \mathbf{U}_{j-1}^k\right). \tag{5.32}$$
- The Lax–Wendroff scheme
$$\begin{aligned}\mathbf{U}_j^{k+1} &= \mathbf{U}_j^k - \frac{\Delta t}{2h} A \left(\mathbf{U}_{j+1}^k - \mathbf{U}_{j-1}^k\right) \\ &\quad + \frac{(\Delta t)^2}{2h^2} A^2 \left(\mathbf{U}_{j-1}^k - 2\mathbf{U}_j^k + \mathbf{U}_{j+1}^k\right). \end{aligned} \tag{5.33}$$

To determine correct boundary conditions, we usually need to find the diagonal form $A = T^{-1}DT$ and the new system $\tilde{\mathbf{u}}_t = D\tilde{\mathbf{u}}_x$ with $\tilde{\mathbf{u}} = T\mathbf{u}$.

5.8 Finite Difference Methods for Conservation Laws

The canonical form for the 1D conservation law is

$$\mathbf{u}_t + \mathbf{f}(u)_x = 0\,, \tag{5.34}$$

and one famous benchmark problem is the Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0\,, \tag{5.35}$$

in which $f(u) = u^2/2$. The term $\mathbf{f}(u)$ is often called the flux. This equation can be written in the nonconservative form

$$u_t + uu_x = 0\,, \tag{5.36}$$

and the solution likely develops shock(s) where the solution is discontinuous,[1] even if the initial condition is arbitrarily differentiable, i.e., $u_0(x) = \sin x$.

We can use the upwind scheme to solve the Burger's equation. From the nonconservative form, we obtain

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + U_j^k \frac{U_j^k - U_{j-1}^k}{h} = 0\,, \quad \text{if } U_j^k \ge 0\,,$$

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + U_j^k \frac{U_{j+1}^k - U_j^k}{h} = 0\,, \quad \text{if } U_j^k < 0\,,$$

or from the conservative form

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{(U_j^k)^2 - (U_{j-1}^k)^2}{2h} = 0\,, \quad \text{if } U_j^k \ge 0\,,$$

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{(U_{j+1}^k)^2 - (U_j^k)^2}{2h} = 0\,, \quad \text{if } U_j^k < 0\,.$$

If the solution is smooth, both methods work well (first-order accurate). However, if shocks develop the conservative form gives much better results than that of the nonconservative form.

We can derive the Lax–Wendroff scheme using the modified equation of the nonconservative form. Since $u_t = -uu_x$,

$$\begin{aligned} u_{tt} &= -u_t u_x - uu_{tx} \\ &= uu_x^2 + u(uu_x)_x \\ &= uu_x^2 + u\left(u_x^2 + uu_{xx}\right) \\ &= 2uu_x^2 + u^2 u_{xx}\,, \end{aligned}$$

so the leading term of the modified equation for the first-order method is

$$u_t + uu_x = \frac{\Delta t}{2}\left(2uu_x^2 + u^2 u_{xx}\right)\,, \tag{5.37}$$

[1] There is no classical solution to the PDE when shocks develop because u_x is not well defined. We need to look for weak solutions.

and the nonconservative Lax–Wendroff scheme for Burger's equation is

$$U_j^{k+1} = U_j^k - \Delta t U_j^k \frac{U_{j+1}^k - U_{j-1}^k}{2h}$$
$$= + \frac{(\Delta t)^2}{2} \left(2U_j^k \left(\frac{U_{j+1}^k - U_{j-1}^k}{2h} \right)^2 + (U_j^k)^2 \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2} \right).$$

5.8.1 Conservative FD Methods for Conservation Laws

Consider the conservation law

$$\mathbf{u}_t + \mathbf{f}(u)_x = 0\,,$$

and let us seek a numerical scheme of the form

$$\mathbf{u}_j^{k+1} = \mathbf{u}_j^k - \frac{\Delta t}{h} \left(\mathbf{g}_{j+\frac{1}{2}}^k - \mathbf{g}_{j-\frac{1}{2}}^k \right), \tag{5.38}$$

where

$$\mathbf{g}_{j+\frac{1}{2}} = \mathbf{g} \left(\mathbf{u}_{j-p+1}^k, \mathbf{u}_{j-p+2}^k, \ldots, \mathbf{u}_{j+q+1}^k \right)$$

is called the numerical flux, satisfying

$$g(u, u, \ldots, u) = f(u)\,. \tag{5.39}$$

Such a scheme is called conservative. For example, we have $g(u) = u^2/2$ for the Burger's equation.

We can derive general criteria that g should satisfy, as follows.

1. Integrate the equation with respect to x from $x_{j-\frac{1}{2}}$ to $x_{j+\frac{1}{2}}$, to get

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t dx = - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(u)_x dx$$
$$= - \left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) \right).$$

2. Integrate the equation above with respect to t from t^k to t^{k+1}, to get

$$\int_{t^k}^{t^{k+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t \, dx \, dt = - \int_{t^k}^{t^{k+1}} \left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) \right) dt\,,$$

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(u(x, t^{k+1}) - u(x, t^k) \right) dx = - \int_{t^k}^{t^{k+1}} \left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) \right) dt\,.$$

Define the average of $u(x,t)$ as

$$\bar{u}_j^k = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t^k)dx\,, \tag{5.40}$$

which is the cell average of $u(x,t)$ over the cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ at the time level k. The expression that we derived earlier can therefore be rewritten as

$$\begin{aligned}
\bar{u}_j^{k+1} &= \bar{u}_j^k - \frac{1}{h}\left(\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt - \int_{t^k}^{t^{k+1}} f(u(x_{j-\frac{1}{2}},t))dt\right)\\
&= \bar{u}_j^k - \frac{\Delta t}{h}\left(\frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt - \frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j-\frac{1}{2}},t))dt\right)\\
&= \bar{u}_j^k - \frac{\Delta t}{h}\left(g_{j+\frac{1}{2}} - g_{j+\frac{1}{2}}\right),
\end{aligned}$$

where

$$g_{j+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt\,.$$

Different conservative schemes can be obtained, if different approximations are used to evaluate the integral.

5.8.2 Some Commonly Used Numerical Scheme for Conservation Laws

Some commonly used schemes are:

- Lax–Friedrichs scheme

$$U_j^{k+1} = \frac{1}{2}\left(U_{j+1}^k + U_{j-1}^k\right) - \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_{j-1}^k)\right); \tag{5.41}$$

- Lax–Wendroff scheme

$$\begin{aligned}
U_j^{k+1} &= U_j^k - \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_{j-1}^k)\right)\\
&\quad + \frac{(\Delta t)^2}{2h^2}\left\{A_{j+\frac{1}{2}}\left(f(U_{j+1}^k) - f(U_j^k)\right) - A_{j-\frac{1}{2}}\left(f(U_j^k) - f(U_{j-1}^k)\right)\right\},
\end{aligned} \tag{5.42}$$

  where $A_{j+\frac{1}{2}} = Df(u(x_{j+\frac{1}{2}},t))$ is the Jacobian matrix of $f(u)$ at $u(x_{j+\frac{1}{2}},t)$.

A modified version

$$\begin{cases} U_{j+\frac{1}{2}}^{k+\frac{1}{2}} = \frac{1}{2}\left(U_j^k + U_{j+1}^k\right) - \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_j^k)\right) \\ U_j^{k+1} = U_j^k - \frac{\Delta t}{h}\left(f(U_{j+\frac{1}{2}}^{k+\frac{1}{2}}) - f(U_{j-\frac{1}{2}}^{k+\frac{1}{2}})\right), \end{cases} \tag{5.43}$$

called the Lax–Wendroff–Richtmyer scheme, does not need the Jacobian matrix.

Exercises

1. Show that the following scheme is consistent with the PDE $u_t + cu_{tx} + au_x = f$:

$$\frac{U_i^{n+1} - U_i^n}{k} + c\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1} - U_{i+1}^n + U_{i-1}^n}{2kh} + a\frac{U_{i+1}^n - U_{i-1}^n}{2h} = f_i^n.$$

   Discuss also the stability, as far as you can.

2. Implement and test the upwind and the Lax–Wendroff schemes for the one-way wave equation

$$u_t + u_x = 0.$$

   Assume the domain is $-1 \le x \le 1$, and $t_{final} = 1$. Test your code for the following parameters:

   (a) $u(t,-1) = 0$, and $u(0,x) = (x+1)e^{-x/2}$.

   (b) $u(t,-1) = 0$, and $u(0,x) = \begin{cases} 0 & \text{if } x < -1/2, \\ 1 & \text{if } -1 \le x \le 1/2, \\ 0 & \text{if } x > 1/2. \end{cases}$

   Do the grid refinement analysis at $t_{final} = 1$ for case (a) where the exact solution is available, take $m = 10, 20, 40$, and 80. For problem (b), use $m = 40$. Plot the solution at $t_{final} = 1$ for both cases.

3. Use the upwind and Lax–Wendroff schemes for Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

   with the same initial and boundary conditions as in problem 2.

4. Solve the following wave equation numerically using a finite difference method.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad 0 < x < 1,$$

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = \begin{cases} x/4 & \text{if } 0 \le x < 1/2, \\ (1-x)/4 & \text{if } 1/2 \le x \le 1. \end{cases}$$

   (a) Test your code and convergence order using a problem that has the exact solution.

   (b) Test your code again by setting $f(x,t) = 0$.

(c) Modify and validate your code to the PDE with a damping term

$$\frac{\partial^2 u}{\partial t^2} - \beta \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x,t).$$

(d) Modify and validate your code to the PDE with an advection term

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + f(x,t),$$

where c and β are positive constants.

5. Download the Clawpack for conservation laws from the Internet. Run a test problem in 2D. Write 2 to 3 pages to detail the package and the numerical results.