1. (a): Let x_e be the solution of $Ax = b$ assuming that $det(A) \neq 0$, $\bar{x}$ be the solution of $Ax = b + \delta b$, show that

$$\frac{\|x_e - \bar{x}\|}{\|x_e\|} \leq cond(A) \frac{\|\delta b\|}{\|b\|}.$$

Hint: $\|b\| = \|AA^{-1}b\| \leq \|A\|\|A^{-1}b\|$.

(b): Given a matrix A that is invertible. Let E is be such a matrix that $\|A^{-1}E\| < \alpha < 1$. Show that $A + E$ is invertible and $\|(A+E)^{-1}\| \leq \dfrac{\|A^{-1}\|}{1-\alpha}$.

2. Plot or sketch the set of $\|\mathbf{x}\|_p = 1$ in two-dimensions, $p = 1, 2, \infty$. This example helps us to understand geometric meanings of vector norms.

3. Given an n-by-n non-singular matrix A, how do you efficiently solve the following problems using Gaussian elimination with partial column pivoting?

   (a) Solve $A^k x = b$, where k is a positive integer.

   (b) Computer $\alpha = c^T A^{-1} b$, where c and b are two vectors.

   You should (1) describe your algorithm; (2) present a pseudo-code; (3) find out the required operation counts.

   Hint: For (a), set $y_{k+1} = b$, $Ay_k = y_{k+1}$. Get $PA = LU$ first, then for j=k, -1, 1, solve $Ay_k = y_{k+1}$, then $y_1 = x$. For (b), let $z = A^{-1}b$ so $Az = b$, there is no need to find A^{-1}.

4. Often a small determinant is a sign of an ill-conditioned matrix. But this exercise is an exception. Given the following matrix A. (a): Find $\|A\|_\infty$. (b): Using Gaussian elimination to find the $A = LU$ decomposition, that is, L and U, and $det(A)$. (c): Let the true solution of $Ax = b$ is $[1,\ 1,\ 1, \cdots, 1]^T$, what is b. Solve the system of equation using computers for $n = 10$, $n = 100$, and $n = 500$ and explain your results. (d): Using $A = LU$ to find $cond_\infty(A)$. Do you think this matrix is ill-conditioned? **Hint: Use Matlab to solve the problem for small n, say, $n = 5, 10, 15, \cdots$ to get the result, then try to prove it. This is an example we can use computer to get theoretical result.**

$$A = \begin{bmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & 1 & & 1 \\ \vdots & \ddots & \ddots & & 1 \\ -1 & -1 & \cdots & -1 & 1 \end{bmatrix}.$$

5. Check whether the following matrices are:

   - Strictly column diagonally dominant.
   - Symmetric positive definite.

Justify your conclusion. What is the significance of knowing these special matrices to the Gaussian related algorithms? Answer this question by considering issues of accuracy, speed, and the storage.

$$\begin{bmatrix} -5 & 2 & 1 & 0 \\ 2 & 7 & -1 & -1 \\ 1 & -1 & 5 & 1 \\ 0 & -1 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} \alpha & \beta \\ -1 & 2 \end{bmatrix}$$

Find the Cholesky decomposition $A = LL^T$ or $A = LDL^T$ for the middle matrix. Note that, you need to determine the range of α and β.

6. Given a matrix A and a vector b

$$A = \begin{bmatrix} \frac{1}{100} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} \frac{1}{100} \\ 1 \\ 1 \end{bmatrix}$$

   (a) Find the condition number of A in 2-norm.

   (b) Given $x_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ and $x_2 = \begin{bmatrix} 1.1 & 1 & 0.9 \end{bmatrix}^T$, find the residual, and the norm of the residual, for both x_1 and x_2 in 2-norm.

   (c) Is one of x_1 and x_2 the exact solution of the system $Ax = b$?

   (d) If x_1 or x_2 is not the solution, use the error estimate discussed in class (relation between the residual and the relative error) to give an estimate error bound for the relative error.

   (e) Find the actual relative error for x_1 and x_2 as approximations to $Ax = b$ and compare with the error bound that you just have got. How much is the difference?

7. (Programming Part) Let A be a symmetric positive definite matrix.

   (a) Derive the algorithms for $A = LDL^T$ decomposition, where L is a unit lower triangular matrix, and D is a diagonal matrix.

   (b) Write a Matlab code (or other language if you prefer) to do the factorization and solve the linear system of equations $Ax = b$ using the factorization. **Hint:** the process is the following:

$$\begin{aligned} Ly &= b, && y \text{ is the unknown,} \\ Dz &= y, && z \text{ is the unknown,} \\ L^T x &= z, && x \text{ is the unknown, which is the solution.} \end{aligned}$$

   Construct at least one example that you know the exact solution to validate your code.