Matrix norms: $A \in R^{m \times n}$, a vector norm is a non-negative function $f(A)$ satisfying

(i) $f(A) \geq 0$ $\quad f(A) = 0$ iff $A = 0$ matrix

(ii): $f(\alpha A) = |\alpha| f(A)$ for any $\alpha \in R$.

(iii) $f(A+B) \leq f(A) + f(B)$

Ex: We can treat $A = \begin{bmatrix} a_1, a_2, \cdots a_n \end{bmatrix} \overset{?}{=} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \tilde{A}$

to get vectorms. $\in R^{m \times n, 1}$

$$\|\tilde{A}\|_2^2 = a_{11}^2 + a_{21}^2 + \cdots + a_{m1}^2 + a_{12}^2 + a_{22}^2 + \cdots + a_{m2}^2$$
$$+ \cdots$$
$$+ a_{1j}^2 + \cdots + a_{mj}^2$$
$$+ a_{1n}^2 + a_{2n}^2 + \cdots + a_{mn}^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}^2$$
$$= \sum_{ij} a_{ij}^2 = \|A\|_F^2$$

Frobenius norm of a matrix $A$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$$

Pros matrix norm

Cons:

$$\|I^{n \times n}\|_F = \sqrt{n} \neq 1$$

$$I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}_{n \times n}$$

Another definition, $\xrightarrow[\text{induced}]{\text{operator}}$ norms

Given two vectorms, $\|x\|_\alpha$, $x \in \mathbb{R}^n$

$\|x\|_\beta$, $x \in \mathbb{R}^m$.

$A \in \mathbb{R}^{m \times n}$, $Ax \in \mathbb{R}^m$

$x \in \mathbb{R}^n$

$$f(A) = \sup_{x \neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha}$$

Check: $f(A) \geq 0$ ✓ $f(A) = 0$, $\sup_{x \neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha} = 0$

$A = 0$, counter proof. If $A \neq 0$, there is at least $a_{ij} \neq 0$, $x = e_j = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ \\ 1 \\ 0 \\ \vdots \end{bmatrix} \end{array}$ $j$

$$Ax = Ae_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix} \neq 0, \quad \|Ax\|_\beta \neq 0,$$

$$\frac{\|Ae_j\|_\beta}{\|e_j\|_\alpha} > 0$$

$\alpha = 1, 2, \infty$ $\|e_j\|_\alpha = 1$.

(ii) $$f(\sigma A) = \sup_{x \neq 0} \frac{\|\sigma A x\|_\beta}{\|x\|_\alpha} \qquad \sup_{x \neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha} > 0$$

$$= \sup_{x \neq 0} \frac{|\sigma| \, \|Ax\|_\beta}{\|x\|_\alpha}$$ $\longrightarrow$ From the vector norm

$$= |\sigma| \sup_{x \neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha} = |\sigma| f(A)$$

(iii) $f(A+B) = \sup_{x\neq 0} \frac{\|(A+B)x\|_\beta}{\|x\|_\alpha} = \sup_{x\neq 0} \frac{\|Ax+Bx\|_\beta}{\|x\|_\alpha}$

$\leq \sup_{x\neq 0} \frac{(\|Ax\|_\beta + \|Bx\|_\beta)}{\|x\|_\alpha}$ vector triangle inequality

$\leq \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|_\alpha} + \sup_{x\neq 0} \frac{\|Bx\|_\beta}{\|x\|_\alpha}$

$= f(A) + f(B)$

$f(A)$ is a matrix norm.

Particularly if $A = I$, $m=n$,

$\|I\| = \sup_{x\neq 0} \frac{\|Ix\|}{\|x\|} = \sup_{x\neq 0} \frac{\|x\|}{\|x\|} = 1$

This norm is called operator induced matrix norm. (can be any space)

associated ✓

Remark, we can replace sup with max.

$\|A\| = \max_{x\neq 0} \frac{\|Ax\|_\alpha}{\|x\|_\beta} \underset{\text{reached}}{\overset{\text{can be}}{=}} \frac{\|Ax^*\|_\alpha}{\|x^*\|_\beta}$

Since $R^n$ is a finite dimensional space $\dim(R^n)=n$

If $m=n$, then $\|A\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x\neq 0} \|A \frac{x}{\|x\|}\|$

The norm of $\frac{x}{\|x\|}$ is number one

$y$ is called a unit vector.

$= \max_{\|y\|=1} \|Ay\|$

Commonly used vector norms $\|x\|_p$, $p=1,2,\infty$

$\|A\|_p$, $p=1,2,\infty$

$$A=\begin{array}{c|c} \begin{bmatrix} -5 & 0 & 7 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix} & \begin{matrix} 12 \\ 5 \\ 1 \end{matrix} \\ \begin{matrix} 6 & 3 & 9 \end{matrix} & \end{array} \qquad \|A\|_\infty = 12$$

$$\|A\|_1 = 9$$

<u>Thm:</u> $\|A\|_\infty = \max\left\{ \sum_{j=1}^{n} |a_{1j}|, \sum_{j=1}^{n} |a_{2j}|, \dots \sum_{j=1}^{n} |a_{ij}|, \dots \sum_{j=1}^{n} |a_{mj}| \right\}$

$A \in \mathbb{R}^{n\times n}$,

$$\|A\|_\infty = \max_{1\le i\le m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} \quad \text{compact form}$$

<u>Proof:</u> We can find $i^*$ such that

$$\max_{1\le i\le m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right. = \sum_{j=1}^{n} |a_{i^*j}| = M$$

<u>Step 1</u>

For any $x \in \mathbb{R}^n$, $\|x\|_\infty = 1$, $\|Ax\|_\infty \le M$

<u>Step 2</u> Find $x^*$ such that $\|Ax^*\|_\infty = M$

$\|x^*\|_\infty = 1$,

<u>Step 1</u>: For any $x$, $\|x\|_\infty = 1$, $\|Ax\|_\infty \le M$

$$Ax = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}$$

$|x_j| \le 1$

$$\left|\sum_{j=1}^{n} a_{kj} x_j\right| \le \sum_{j=1}^{n} |a_{kj}|\,|x_j| \le \sum_{j=1}^{n} |a_{kj}| \le \sum_{j=1}^{n} |a_{i^*,j}| = M$$

$\|Ax\|_\infty \le M$

<u>Step 2</u> $x^*$, $\|x^*\|_\infty = 1$, $\|Ax^*\|_\infty = M$.

$$Ax^* = \begin{bmatrix} \sum a_{1j} x_j^* \\ \vdots \\ \sum a_{i^*,j} x_j^* \end{bmatrix} = \begin{bmatrix} * \\ \vdots \\ \sum_{j=1}^{n} |a_{i^*,j}| \end{bmatrix} = \begin{bmatrix} \le M \\ \le M \\ M \\ \le M \end{bmatrix}$$

$$x^* = \begin{bmatrix} \operatorname{sgn}(a_{i^*,1}) \\ \operatorname{sgn}(a_{i^*,2}) \\ \\ \operatorname{sgn}(a_{i^*,n}) \end{bmatrix}$$

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

$\operatorname{sign}(x)$

$$Ax^* = \begin{bmatrix} * \\ \vdots \\ a_{i^*,1}\operatorname{sgn}(a_{i^*,1}), \quad a_{i^*,2}\operatorname{sgn}(a_{i^*,2}) \quad \cdots \quad a_{i^*,n}\operatorname{sgn}(a_{i^*,n}) \end{bmatrix}$$

$$= \begin{bmatrix} * \\ * \\ \sum_{j=1}^{n} |a_{i^*,j}| \end{bmatrix} \qquad \|Ax^*\|_\infty = M$$

<u>Thm:</u> $\|A\|_1 = \max\left\{ \sum_{i=1}^{m} |a_{i,1}|, \sum_{i=1}^{m} |a_{i,2}|, \dots, \sum_{i=1}^{m} |a_{i,n}| \right\}$

$$= \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}| .$$

<u>Thm:</u> $\|A\|_2$,

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \max_{\|x\|_2=1} \sqrt{(Ax, Ax)}$$

$$= \max_{\|x\|_2=1} (Ax)^T Ax$$

$$= \max_{\|x\|_2=1} \sqrt{x^T (A^T A) x}$$

Schur matrix decomposition theorem.

If $B \in \mathbb{R}^{n \times n}$ is symmetric, $B^T = B$, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & \end{bmatrix}$

there is an orthogonal matrix (unitary matrix)

$U$, $U^T U = I$, $U U^T = I$ such that

$$U^T B U = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$\lambda_i$, $i = 1, 2, \dots n$ are the eigenvalue of $B$.

Is $U$ invertible? yes. $U^{-1} = U^T$, $\det(U)$

$$1 = \det(I) = \det(U U^T) = \det(U) \cdot \det(U^T) = (\det(U))^2$$

$$\det(U) = \pm 1$$

$$\|A\|_2 = \max_{1 \le i \le n} \sqrt{\lambda_i (A^T A)} , \text{ why ?}$$

Challenge:

1. True or False: If $\|A\|_\infty = \|A\|_1$, then $A = A^T$. False, the reverse is true.

A ~~necessary~~ condition. sufficient.

2. Find $\|A\|_p$, $p = 1, 2, \infty$

$$A = \begin{array}{cc} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} & \begin{array}{c} 3 \\ 3 \end{array} \\ \begin{array}{cc} 3 & 3 \end{array} & \end{array}, \qquad A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Skew-symmetric

$a_{ij} = -a_{ji}$ $i \neq j$ $\qquad \|A\|_\infty = \|A\|_1 = 3$

$$\|A\|_2 = \sqrt{\max_{1\le i\le m} \lambda_i(AA^T)}$$

$$= \sqrt{\max_{1\le i\le n} \lambda_i(A^TA)}$$

$A \in \mathbb{R}^{m\times n}$

$AA^T \in \mathbb{R}^{m\times m}$

$A^TA \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$\lambda_1(A^TA) = 5$

$\lambda_2(A^TA) = 5$

$\|A\|_2 = \sqrt{5} = 2.$

$\|A\|_\infty = 3$

$\|A\|_1 = 3$

$\|x\|_\infty \le \|x\|_2 \le \|x\|_2$

Prove $\|A\|_2 = \sqrt{\max_{1\le i\le n} A^TA}$ $\quad A \in R^{n\times n}$.

$A^TA$ is a symmetric matrix, from Schur thm., $\exists\, U$, $U^HU = UU^H = I$, such that

$$U^TAU = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \quad \text{or} \quad A = UDU^H$$

$U = \{u_{ij}\}$, $\quad U^H = \{\bar{u}_{ji}\}$, $\quad u_{ij} = \alpha_{ij} + i\beta_{ij}$

$\bar{u}_{ji} = \alpha_{ji} - i\beta_{ji}$

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

$$= \max_{\|x\|_2=1} \|UDU^Hx\|_2 = \max_{\|x\|_2=1} \sqrt{x^T\underline{UD}\underline{U^HU}DU^Hx}$$

$$= \max_{\|x\|_2=1} \sqrt{(U^Hx)^H D^2 (U^Hx)} \qquad U^Hx = y$$

$$= \max_{\|x\|_2=1} \sqrt{y^H D^2 y} = \max \sum_{i=1}^n \lambda_i^2 y_i^2$$

$$\|y\|_2 = \sqrt{y^Ty} = \sqrt{(U^Hx)^H U^Hx} = \sqrt{x^H \underline{\underline{UU^H}} x} = \sqrt{x^Hx} = \|x\|_2 = 1$$

$\sum y_i^2 = 1$, $\quad |y_i| \le 1$.

$$\|Ax\|_2 \le \sqrt{\sum_{i=1}^n \lambda_{max}^2 y_i^2} = \sqrt{\lambda_{max}(A^TA) \sum_{i=1}^n y_i^2}$$

$$\le \lambda_{max}(A^TA)$$

<u>Step 2</u>: Find the $x^*$ such that $\|x^*\|_2 = 1$

$$\|Ax^*\| = \lambda_{max}(A^TA)$$

$$A^T A x^* = \lambda_{max} x^*, \quad \|x^*\|_2 = 1.$$

$$(x^*)^T A^T A x^* = \lambda_{max} (x^*)^T x^*$$

$$\|A x^*\|_2^2 = \lambda_{max}, \quad \|A x^*\|_2 = \sqrt{\lambda_{max}}$$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \|A\|_1 = \|A\|_\infty = 3$$

$$A = A^T, \quad \lambda_i(A^T A) = \lambda_i(A^2) = (\lambda_i(A))^2$$

Thm: If $A = A^T$, then $\|A\|_2 = \max_{1 \le i \le n} |\lambda_i(A)|$

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{pmatrix} = 0$$

$$(\lambda-2)^2 - 1^2 = 0 \quad (\lambda-2)^2 = 1$$

$$\lambda - 2 = \begin{cases} 1 & \lambda_1 = 3 \\ -1 & \lambda_2 = 1 \end{cases}$$

$$\|A\|_2 = 3$$

<u>Thm:</u> All matrix norms are equivalent.

$$? \|A\|_1 \le \|A\|_2 \le ? \|A\|_1$$

In finite dimension. which is not true for $L^2(a,b)$

How to solve $Ax=b$, $A \in \mathbb{R}^{n\times n}$, $\det(A) \neq 0$.

efficiently
- accurate
- fast
- least storage.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$x_1 = \frac{\det\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}}{\det(A)}, \quad x_2 =$$

Solving $Ax=b$ has the same complexity as finding $\det(A)$. Cram method is not efficient.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ 0 & & \ddots & \\ & & & a_{nn} \end{bmatrix},$$ A is an upper triangular matrix

$\det(A) \neq 0$, $a_{ii} \neq 0$, $\prod_{j=1}^{n} a_{jj} \neq 0$

1-st eqn. $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$x_n = b_n / a_{nn}$

$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$ ✓ $a_{nn}x_n = b_n$

$x_{n-1} = (b_{n-1} - a_{n-1,n}x_n) / a_{n-1,n-1}$

The backward substitution. Pseudo-code

$x_n = b_n / a_{nn}$.
for $i = n-1, -1, 1$
$x_i = \left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j\right) / a_{ii}$ → Expand
end

$x_i = b_i$ init.
for $j = i+1 : n$
$x_i = x_i - a_{ij} x_j$
end
$x_i = x_i / a_{ii}$

Complexity analysis, operation counts

$n-(i+1)+1+1$ $\times, \div$

$n-(i+1)+1+1$ $+ -$

$i=1$, $n$

$i=2$ $n-1$

$i=n$, $1$

Total $\times$ and $\div$ $1+2+3+\cdots+n$

$= \sum_{j=1}^{n} j = \frac{1+n}{2} \cdot n$

$= O(n^2/2)$ ✓

$+/-$ $O(n^2/2)$

In general $A \longrightarrow U$ without changing solution.

Idea: Gaussian elimination process to transform $Ax=b$ to $Ux=\tilde{b}$

$$\overset{A}{\left[\begin{array}{cccc|c} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{array}\right]} \xrightarrow{\text{1-st}} \left[\begin{array}{cccc|c} \times & \times & \times & \times & \times \\ 0 & \otimes & \otimes & \otimes & \otimes \\ 0 & \otimes & \otimes & \otimes & \otimes \\ 0 & \otimes & \otimes & \otimes & \otimes \end{array}\right]$$

$$\xrightarrow{\text{2nd}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right] \xrightarrow{\text{3rd}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 & \cdots & 0 \\ \otimes & 0 & 1 & & \\ & 0 & \otimes & \ddots & \\ -\frac{a_{n1}}{a_{11}} & \vdots & & & 1 \end{bmatrix} \overset{A^{(1)}}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & a_{1\,n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & a_{2\,n+1} \\ & & & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \vdots & a_{n\,n+1} \end{bmatrix}}$$

$$Ax = b$$

$$\underline{L_1 A' x = L_1 b'}$$

$$A^{(2)} x = b^{(2)}$$

$$= \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a & \cdots & a_{1n} & a_{1\,n+1} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & a_{2\,n+1}^{(2)} \\ 0 & & & & & \\ \vdots & & a_{ij}^{(2)} & & & \\ 0 & & & & & \end{array}\right]$$

$$-\frac{a_{21}}{a_{11}} a_{11} + a_{21}$$

$$-\frac{a_{31}}{a_{11}} a_{11} + a_{31}$$

Use ith row times jth column

$$\boxed{a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i1}^{(1)}}{a_{11}^{(1)}} a_{1j}^{(1)}}$$

$$a_{ij}^{(2)} = -\frac{a_{i1}}{a_{11}} a_{1j} + 1 \cdot a_{ij}$$

$i = 2, 3, \ldots n, \quad j = 2, 3, \ldots, n, n+1.$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ -\frac{a_{32}^{(2)}}{a_{22}^{(2)}} & & 1 & \\ & & & \ddots \end{bmatrix} \left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & a_{1\,n+1}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & a_{2\,n+1}^{(2)} \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right] = \begin{bmatrix} a_{11}^{(1)} & \\ 0 & a_{22}^{(2)} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \quad a_{ij}^{(3)}$$

$$-\frac{a_{42}^{(2)}}{a_{22}^{(2)}}$$

$$a_{ij}^{(3)} = a_{ij}^{(2)} - \frac{a_{i2}^{(2)}}{a_{22}^{(2)}} a_{2j}^{(2)}$$

$i = 3, 4, \ldots, n$

$j = 3, 4, \ldots, n, n+1$

```
For k = 1, 2, ... n-1.
    for i = k+1, n
        for j = k+1, n+1
            a_ij^(k+1) = a_ij^(k) - a_ik^(k) a_kj^(k) / a_kk^(k)
        end
    end
end
```