Cylindrical co-ordinates

To the co-ordinates $\xi_1 = x$, $\xi_2 = \sigma$, $\xi_3 = \phi$ (where ϕ is the azimuthal angle about the axis $\sigma = 0$) there correspond the scale factors

$$h_1 = 1, \quad h_2 = 1, \quad h_3 = \sigma.$$

Then
$$\frac{\partial \mathbf{a}}{\partial \phi} = 0, \quad \frac{\partial \mathbf{b}}{\partial \phi} = \mathbf{c}, \quad \frac{\partial \mathbf{c}}{\partial \phi} = -\mathbf{b},$$

and **a**, **b**, **c** are independent of x and σ.

$$\nabla V = \mathbf{a}\frac{\partial V}{\partial x} + \mathbf{b}\frac{\partial V}{\partial \sigma} + \frac{\mathbf{c}}{\sigma}\frac{\partial V}{\partial \phi},$$

$$\mathbf{n}.\nabla\mathbf{F} = \mathbf{a}(\mathbf{n}.\nabla F_x) + \mathbf{b}\left(\mathbf{n}.\nabla F_\sigma - \frac{n_\phi F_\phi}{\sigma}\right) + \mathbf{c}\left(\mathbf{n}.\nabla F_\phi + \frac{n_\phi F_\sigma}{\sigma}\right),$$

$$\nabla.\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{1}{\sigma}\frac{\partial(\sigma F_\sigma)}{\partial \sigma} + \frac{1}{\sigma}\frac{\partial F_\phi}{\partial \phi},$$

$$\nabla\times\mathbf{F} = \mathbf{a}\left\{\frac{1}{\sigma}\frac{\partial(\sigma F_\phi)}{\partial \sigma} - \frac{1}{\sigma}\frac{\partial F_\sigma}{\partial \phi}\right\} + \mathbf{b}\left(\frac{1}{\sigma}\frac{\partial F_x}{\partial \phi} - \frac{\partial F_\phi}{\partial x}\right) + \mathbf{c}\left(\frac{\partial F_\sigma}{\partial x} - \frac{\partial F_x}{\partial \sigma}\right),$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma\frac{\partial V}{\partial \sigma}\right) + \frac{1}{\sigma^2}\frac{\partial^2 V}{\partial \phi^2},$$

$$\nabla^2\mathbf{F} = \mathbf{a}(\nabla^2 F_x) + \mathbf{b}\left(\nabla^2 F_\sigma - \frac{F_\sigma}{\sigma^2} - \frac{2}{\sigma^2}\frac{\partial F_\phi}{\partial \phi}\right) + \mathbf{c}\left(\nabla^2 F_\phi + \frac{2}{\sigma^2}\frac{\partial F_\sigma}{\partial \phi} - \frac{F_\phi}{\sigma^2}\right).$$

Rate-of-strain tensor:

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{\sigma\sigma} = \frac{\partial u_\sigma}{\partial \sigma}, \quad e_{\phi\phi} = \frac{1}{\sigma}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\sigma}{\sigma},$$

$$e_{\sigma\phi} = \frac{\sigma}{2}\frac{\partial}{\partial \sigma}\left(\frac{u_\phi}{\sigma}\right) + \frac{1}{2\sigma}\frac{\partial u_\sigma}{\partial \phi}, \quad e_{\phi x} = \frac{1}{2\sigma}\frac{\partial u_x}{\partial \phi} + \frac{1}{2}\frac{\partial u_\phi}{\partial x}, \quad e_{x\sigma} = \frac{1}{2}\frac{\partial u_\sigma}{\partial x} + \frac{1}{2}\frac{\partial u_x}{\partial \sigma}.$$

Equation of motion for an incompressible fluid, with no body force:

$$\frac{\partial u_x}{\partial t} + \mathbf{u}.\nabla u_x = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u_x,$$

$$\frac{\partial u_\sigma}{\partial t} + \mathbf{u}.\nabla u_\sigma - \frac{u_\phi^2}{\sigma} = -\frac{1}{\rho}\frac{\partial p}{\partial \sigma} + \nu\left(\nabla^2 u_\sigma - \frac{u_\sigma}{\sigma^2} - \frac{2}{\sigma^2}\frac{\partial u_\phi}{\partial \phi}\right),$$

$$\frac{\partial u_\phi}{\partial t} + \mathbf{u}.\nabla u_\phi + \frac{u_\sigma u_\phi}{\sigma} = -\frac{1}{\rho\sigma}\frac{\partial p}{\partial \phi} + \nu\left(\nabla^2 u_\phi + \frac{2}{\sigma^2}\frac{\partial u_\sigma}{\partial \phi} - \frac{u_\phi}{\sigma^2}\right).$$

Polar co-ordinates in two dimensions

The relevant formulae can be obtained from those for the above cylindrical co-ordinates by suppressing all components and derivatives in the direction

of the x-co-ordinate line, but are written out here in view of the frequency of their use. The co-ordinates are

$$\xi_1 = r, \quad \xi_2 = \theta, \quad \text{and} \quad h_1 = 1, \quad h_2 = r,$$

$$\frac{\partial \mathbf{a}}{\partial r} = 0, \quad \frac{\partial \mathbf{a}}{\partial \theta} = \mathbf{b}, \quad \frac{\partial \mathbf{b}}{\partial r} = 0, \quad \frac{\partial \mathbf{b}}{\partial \theta} = -\mathbf{a}.$$

$$\nabla V = \mathbf{a}\frac{\partial V}{\partial r} + \frac{\mathbf{b}}{r}\frac{\partial V}{\partial \theta},$$

$$\mathbf{n}.\nabla\mathbf{F} = \mathbf{a}\left(\mathbf{n}.\nabla F_r - \frac{n_\theta F_\theta}{r}\right) + \mathbf{b}\left(\mathbf{n}.\nabla F_\theta + \frac{n_\theta F_r}{r}\right),$$

$$\nabla.\mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta},$$

$$\nabla\times\mathbf{F} = \left\{\frac{1}{r}\frac{\partial(rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right\}\mathbf{a}\times\mathbf{b},$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2},$$

$$\nabla^2\mathbf{F} = \mathbf{a}\left(\nabla^2 F_r - \frac{F_r}{r^2} - \frac{2}{r^2}\frac{\partial F_\theta}{\partial \theta}\right) + \mathbf{b}\left(\nabla^2 F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial \theta} - \frac{F_\theta}{r^2}\right).$$

Rate-of-strain tensor:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta}.$$

Equation of motion for an incompressible fluid, with no body force:

$$\frac{\partial u_r}{\partial t} + \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}\right)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right).$$

when finding the components of $\nabla^2\mathbf{F}$ in a particular co-ordinate system, to use the identity
$$\nabla^2\mathbf{F} = \nabla(\nabla.\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F})$$
and the above expressions for grad, div and curl.

Consider now the components of the rate-of-strain tensor expressed in terms of velocity components and derivatives relative to the curvilinear system. The gradient, in the direction $\mathbf{n}$, of the component of velocity $\mathbf{u}$ in the fixed direction $\mathbf{m}$ is
$$\mathbf{n}.\nabla(\mathbf{m}.\mathbf{u}), \quad = \mathbf{m}.(\mathbf{n}.\nabla\mathbf{u}).$$
Diagonal elements of the rate-of-strain tensor represent rates of extension, obtained by putting $\mathbf{m} = \mathbf{n}$, and the non-diagonal elements involve velocity gradients for which $\mathbf{m}$ and $\mathbf{n}$ are orthogonal. We see then, from the above formula for $\mathbf{n}.\nabla\mathbf{F}$, that the components of the rate-of-strain tensor relative to Cartesian axes locally parallel to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (to which the suffixes 1, 2, 3 refer, respectively) are
$$e_{11} = \mathbf{a}.(\mathbf{a}.\nabla\mathbf{u}) = \frac{1}{h_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{u_3}{h_3 h_1}\frac{\partial h_1}{\partial \xi_3},$$
$$e_{23} = \tfrac{1}{2}\mathbf{b}.(\mathbf{c}.\nabla\mathbf{u}) + \tfrac{1}{2}\mathbf{c}.(\mathbf{b}.\nabla\mathbf{u}) = \frac{h_3}{2h_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_3}{h_3}\right) + \frac{h_2}{2h_3}\frac{\partial}{\partial \xi_3}\left(\frac{u_2}{h_2}\right),$$
with four other expressions obtained by cyclic interchange of suffixes. The components of the stress tensor σ_{ij} can be obtained from those of rate of strain, using the relation (for an incompressible fluid)
$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}.$$
The components of all terms in the equation of motion of a fluid in the directions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ may now be found by simple substitution in the appropriate expressions above. The components of the term $\mathbf{u}.\nabla\mathbf{u}$ in the acceleration are obtained from the expression for $\mathbf{n}.\nabla\mathbf{F}$.

Applications to some particular co-ordinate systems are as follows.

Spherical polar co-ordinates

To the co-ordinates $\xi_1 = r$, $\xi_2 = \theta$, $\xi_3 = \phi$ (where ϕ is the azimuthal angle about the axis $\theta = 0$) there correspond the scale factors
$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta. \qquad 0 \le \varphi \le 2\pi$$
$$0 \le \theta \le \pi$$
Then
$$\frac{\partial\mathbf{a}}{\partial r} = 0, \quad \frac{\partial\mathbf{a}}{\partial\theta} = \mathbf{b}, \quad \frac{\partial\mathbf{a}}{\partial\phi} = \sin\theta\,\mathbf{c},$$
$$\frac{\partial\mathbf{b}}{\partial r} = 0, \quad \frac{\partial\mathbf{b}}{\partial\theta} = -\mathbf{a}, \quad \frac{\partial\mathbf{b}}{\partial\phi} = \cos\theta\,\mathbf{c},$$
$$\frac{\partial\mathbf{c}}{\partial r} = 0, \quad \frac{\partial\mathbf{c}}{\partial\theta} = 0, \quad \frac{\partial\mathbf{c}}{\partial\phi} = -\sin\theta\,\mathbf{a} - \cos\theta\,\mathbf{b}.$$
$$\nabla V = \mathbf{a}\frac{\partial V}{\partial r} + \frac{\mathbf{b}}{r}\frac{\partial V}{\partial\theta} + \frac{\mathbf{c}}{r\sin\theta}\frac{\partial V}{\partial\phi},$$

$$\mathbf{n}.\nabla\mathbf{F} = \mathbf{a}\left(\mathbf{n}.\nabla F_r - \frac{n_\theta F_\theta}{r} - \frac{n_\phi F_\phi}{r}\right) + \mathbf{b}\left(\mathbf{n}.\nabla F_\theta - \frac{n_\phi F_\phi}{r}\cot\theta + \frac{n_\theta F_r}{r}\right)$$
$$+\mathbf{c}\left(\mathbf{n}.\nabla F_\phi + \frac{n_\phi F_r}{r} + \frac{n_\phi F_\theta}{r}\cot\theta\right),$$
$$\nabla.\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi},$$
$$\nabla\times\mathbf{F} = \frac{\mathbf{a}}{r\sin\theta}\left\{\frac{\partial(F_\phi\sin\theta)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right\} + \frac{\mathbf{b}}{r}\left\{\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(rF_\phi)}{\partial r}\right\} + \frac{\mathbf{c}}{r}\left\{\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right\},$$
$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2},$$
$$\nabla^2\mathbf{F} = \mathbf{a}\left\{\nabla^2F_r - \frac{2F_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial\phi}\right\}$$
$$+\mathbf{b}\left\{\nabla^2F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial\theta} - \frac{F_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial\phi}\right\}$$
$$+\mathbf{c}\left\{\nabla^2F_\phi + \frac{2}{r^2\sin\theta}\frac{\partial F_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\phi} - \frac{F_\phi}{r^2\sin^2\theta}\right\}.$$
Rate-of-strain tensor:
$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \quad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r},$$
$$e_{\theta\phi} = \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial\phi}, \quad e_{\phi r} = \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right),$$
$$e_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta}.$$
Equation of motion for an incompressible fluid, with no body force:
$$\frac{\partial u_r}{\partial t} + \mathbf{u}.\nabla u_r - \frac{u_\theta^2}{r} - \frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\nu\left\{\nabla^2u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right\},$$
$$\frac{\partial u_\theta}{\partial t} + \mathbf{u}.\nabla u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}$$
$$+\nu\left\{\nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right\},$$
$$\frac{\partial u_\phi}{\partial t} + \mathbf{u}.\nabla u_\phi + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi\cot\theta}{r} = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi}$$
$$+\nu\left\{\nabla^2u_\phi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r^2\sin^2\theta}\right\}.$$