Summary: Orthogonal Functions

1. Let $C^0(a,b)$ denote the space of continuous functions on the interval [a,b]. Using the fact that the sum of continuous functions is continuous and the product of a continuous function with a scalar is again a continuous function, one can show that this space is a vector space. Another vector space of interest is the space $C^2(a,b)$ of all twice differentiable functions on the interval [a,b].

2. In 3-dimensional space one introduces the "dot product" to define such things as: **norm of a vector**, **inner product of two vectors**, **angle between two vectors**, **orthogonality of two vectors**, and more. We can also introduce an inner product in the space $C^0(a,b)$ as follows:

   DEF: The inner product of two vectors $f, g \in C^0(a,b)$ is the number

$$(f,g) = \int_a^b f(x)g(x)dx$$

3. Using this inner product we can make a few definitions patterned after the definitions we know from 3-dimensional space.

   DEF: Two vectors $f, g \in C^0(a,b)$ are **orthogonal** if $(f,g) = 0$.

   DEF: The **norm** of a vector $f \in C^0(a,b)$ is the number $|| \ f \ || = \sqrt{(f,f)}$.

   DEF: A infinite collection of vectors $\mathcal{B} = \{ \ f_1, f_2, f_3, \ldots \}$ is an **orthogonal set** if

   (a) The norm $|| \ f_i \ ||$ of each vector $f_i \in \mathcal{B}$ in the collection is non-zero for all $i = 1, 2, 3, \ldots$.

   (b) The vectors are mutually orthogonal. That is too say, $(f_i, f_j) = 0$ for $i \neq j$, for all $i, j = 1, 2, 3, \ldots$.

   Furthermore, if each vector $f_i \in \mathcal{B}$ has norm 1, then the collection $\mathcal{B}$ is an **orthonormal set** of vectors.

4. It turns out that we must generalize the above definition slightly in order to handle certain cases that we will study. Here is the generalization.

   DEF: Let $w(x)$ we a non-negative-valued, piecewise continuous function that is not identically zero on any subinterval of $[a,b]$. $w(x)$ is called a **weight function**. The inner product **with respect to the weight** $w(x)$ of two vectors $f, g \in C^0(a,b)$ is the number

$$(f,g) = \int_a^b w(x)f(x)g(x)dx$$

   All of the definitions above in 3 have generalizations where the weight function $w(x)$ is included in the integrand in the definition, and we add the words **with respect to the weight** $w(x)$ to each definition. For example:

   DEF: Two vectors $f, g \in C^0(a,b)$ are **orthogonal with respect to the weight** $w(x)$ if

$$(f,g) = \int_a^b w(x)f(x)g(x)dx = 0$$

   DEF: The norm of $f \in C^0(a,b)$ **with respect to the weight** $w(x)$ is

$$|| \ f \ || = \sqrt{\int_a^b w(x)f(x)^2 dx}$$

Regular, Periodic and Singular Sturm-Liouville Problems

In the three problems listed below we make the following assumptions. We assume that the functions $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous on the interval $a \le x \le b$, and that $p(x) > 0$ and $r(x) > 0$ on the open interval $a < x < b$.

1. **A Regular SL - Problem on [a,b]:** We assume additionally that $p(x) > 0$ and $r(x) > 0$ at the endpoints of the interval.

   Find numbers λ for which there are non-trivial solution of the ODE
   $$(p(x)y')' + (q(x) + \lambda r(x))y(x) = 0$$
   and the regular boundary conditions
   $$a_1 y(a) + a_2 y'(a) = 0 \qquad a_1^2 + a_2^2 \neq 0$$
   $$b_1 y(b) + b_2 y'(b) = 0 \qquad b_1^2 + b_2^2 \neq 0$$

   Example: $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) = 0$.

2. **A Periodic SL - Problem on [a,b]:** We assume additionally that $p(a) = p(b)$.

   Find numbers λ for which there are non-trivial solution of the ODE
   $$(p(x)y')' + (q(x) + \lambda r(x))y(x) = 0$$
   and the periodic boundary conditions
   $$y(a) = y(b)$$
   $$y'(a) = y'(b)$$

   Example: $y'' + \lambda y = 0$, $y(-\pi) = y(\pi)$, $y'(-\pi) = y'(\pi)$.

3. **A Singular SL - Problem on [a,b]:** Find numbers λ for which there are non-trivial solution of the ODE
   $$(p(x)y')' + (q(x) + \lambda r(x))y(x) = 0$$
   and one of the following three types of boundary conditions:

   (a) TYPE I: $p(a) = 0$ and
   $$b_1 y(b) + b_2 y'(b) = 0 \qquad b_1^2 + b_2^2 \neq 0$$

   (b) TYPE II: $p(b) = 0$ and
   $$a_1 y(a) + a_2 y'(a) = 0 \qquad a_1^2 + a_2^2 \neq 0$$

   (c) TYPE III: $p(a) = p(b) = 0$. Then there are no specified boundary conditions, but the solutions are required to be **bounded** functions on the interval $[a, b]$.

   Example for TYPE III: $((1 - x^2)y')' + \lambda y = 0$, $-1 \le x \le 1$. When $\lambda = n(n + 1)$, $n = 0, 1, 2, \ldots$, then the ODE is Legendre's differential equation.

4. **A Singular SL - Problem on an infinite interval $(-\infty, \infty)$, (a, ∞) or $(-\infty, b)$:**

5. **Terminology:** For any of the SL - Problems, a number λ such that a non-trivial solution exists to the ODE and boundary conditions is called an **eigenvalue** (e-value) of the problem, and a corresponding solution is called an **eigenvector** (e-vector). Note the non-trivial requirement to be an e-vector.