I A theorem on linear homogeneous 2nd order ODE

Theorem Consider the linear second order homogenous ODE

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad (**)$$

where $a(x)$, $b(x)$ and $c(x)$ are continuous on some interval I, and $a(x)$ never vanishes on I.

(a) (The representation theorem) Let y_1 and y_2 be two linearly independent solutions on I of $(**)$. Then every solution y on I of $(**)$ can be represented in the form

$$y = c_1y_1 + c_2y_2$$

for some constants c_1 and c_2.

(b) (The linear independence theorem) Two solutions y_1 and y_2 on I of $(**)$ are linearly independent on I if and only if the Wronskian $W(y_1, y_2)(x)$ of y_1 and y_2 defined below is not zero for some $x \in I$.

$$W(y_1, y_2)(x) = det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$$

II Constant coefficients

Assume that $a(x)$, $b(x)$ and $c(x)$ in $(**)$ are **all constants**, with $a \neq 0$, so the ODE is now in the form

$$ay'' + by' + cy = 0 \qquad (\ddagger)$$

Substituting $y = e^{rx}$ into $(\ddagger)$ above and canceling the common factor e^{rx} we obtain the *characteristic equation* of the ODE $(\ddagger)$:

$$\boxed{ar^2 + br + c = 0}$$

The solutions of this equation are given by the quadratic equation: $\boxed{r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$.

Case I $b^2 - 4ac > 0$. Then we have two real distinct roots r_1 and r_2, with $r_1 \neq r_2$, and two linearly independent solutions on R of $(\ddagger)$ are $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$. The general solution in this case is

$$\boxed{y = c_1e^{r_1x} + c_2e^{r_2x}}$$

Case II $b^2 - 4ac = 0$. Then we have a repeated root $r_1 = r_2$. Two linearly independent solutions on R of $(\ddagger)$ are $y_1 = e^{r_1x}$ and $y_2 = xe^{r_1x}$. The general solution in this case is

$$\boxed{y = (c_1 + c_2x)e^{r_1x}}$$

Case III $b^2 - 4ac < 0$. Then we have a pair of complex conjugate roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. Two linearly independent solutions on R of $(\ddagger)$ are $y_1 = e^{\alpha x}cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$. The general solution in this case is

$$\boxed{y = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))}$$

III The non-homogeneous problem

Theorem (The representation theorem) Consider the linear second order non-homogenous ODE

$$a(x)y'' + b(x)y' + c(x)y = f(x) \qquad (**\ddagger)$$

where $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are continuous on some interval I, and $a(x)$ never vanishes on I. Let y_1 and y_2 be two linearly independent solutions on I of the associated homogeneous equation $(**)$ above, and let y_p be a *particular solution*[1] of the non-homogeneous equation $(**\ddagger)$. Then every solution y on I of $(**\ddagger)$ can be represented in the form

$$y = c_1 y_1 + c_2 y_2 + y_p$$

for some constants c_1 and c_2.

IV Finding particular solutions

(a) Method of undetermined coefficients for the case of constant coefficients

See the textbook for the general case.

(b) Method of variation of parameters for the general case

Let y_1 and y_2 be two linearly independent solutions on I of the homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ where $a(x)$, $b(x)$ and $c(x)$ are continuous on some interval I, and $a(x)$ never vanishes on I. Then by division by $a(x)$ we can put the equation into the form

$$\boxed{y'' + p(x)y' + q(x)y = g(x)}$$

where $p(x)$, $q(x)$ and $g(x)$ are continuous on I. The method of variation of parameters yields a particular solution to this non-homogeneous equation.

NOTE: The formulas below required that the ODE be put in the above form, where the coefficient of y'' is 1.

We assume a solution of the form $y_p = u_1(x)y_1 + u_2(x)y_2$. The formulas for u_1 and u_2 are

$$u_1 = \int \frac{-y_2 g(x)}{W(y_1, y_2)} dx \qquad\qquad u_2 = \int \frac{y_1 g(x)}{W(y_1, y_2)} dx$$

where the convention is that we choose 0 for the additive constants. Then the particular solution is given by

$$\boxed{y_p = y_1 \left(\int \tfrac{-y_2 g(x)}{W(y_1, y_2)} dx \right) + y_2 \left(\int \tfrac{y_1 g(x)}{W(y_1, y_2)} dx \right)}$$

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[1] i.e. y_p contains no arbitrary constants