Linear Algebra Review

Matrices and Systems of Linear Equations

Matrix: rectangular array of numbers arranged in rows, columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})$$

row index, col. index

m rows
n columns
$m \times n$ matrix — order of A

$n \times n$ matrix called square

If $m = 1$ (one row), called row vector.

If $n = 1$ (one column), called column vector

Vectors: short for column vectors, e.g.,

$$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

m-vector n-vector

Matrix Multiplication

A $m \times n$ B $n \times p$

$C = AB$ is $m \times p$ $\qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

= inner product of i^{th} row of A with j^{th} column of B

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 3 \\ -4 & 2 \\ 7 & 1 \end{bmatrix}$$

$1 \cdot 6 + 0 \cdot 2 + 1 \cdot 1 = 7$

System of m linear equations in n unknowns

$$\begin{aligned} a_{11} x_1 + \cdots + a_{1n} x_n &= b_1 \\ &\vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n &= b_m \end{aligned}$$

Written in matrix form

$$A \underset{\sim}{x} = \underset{\sim}{b}$$

Matrix multiplication is associative, not necessarily commutative.

Diagonal and Triangular Matrices

Suppose A is square, $n \times n$

$a_{11}, a_{22}, \cdots, a_{nn}$ called diagonal entries of A

a_{ij} with $i \neq j$ called off-diagonal entries

a_{ij} with $i < j$ called super-diagonal entries

a_{ij} with $i > j$ called sub-diagonal entries

If $a_{ij} = 0$ for $i \neq j$, A called diagonal matrix

If $a_{ij} = 0$ for $i > j$, A called upper-triangular

If $a_{ij} = 0$ for $i < j$, A called lower-triangular

Exercise: Show that a square matrix is diagonal iff it is both upper- and lower-triang.

Identity Matrix

I_n $n \times n$ (If n is understood, just write I.)

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$I_n A = A$ ($\forall n \times p$ matrices A)

$B I_n = B$ ($\forall m \times n$ matrices B)

Inverse matrices

Square $n \times n$ matrix A called invertible if $\exists$ an $n \times n$ matrix B such that

$$AB = BA = I_n \quad ; \quad B \text{ is called the inverse of } A \text{, written } A^{-1}.$$

Exercise: Show that A can have at most one inverse.

Facts: If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.

If A, B invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$

Matrix addition: A, B same order $\quad A + B = C \quad c_{ij} = a_{ij} + b_{ij}$

Scalar multiplication: $(\alpha A)_{ij} = \alpha a_{ij}$ $\quad$ (α scalar)

Properties:

$A+B = B+A$ $\qquad (A+B)C = AC + BC$

$(A+B)+C = A+(B+C)$ $\qquad \alpha(AB) = (\alpha A)B = A(\alpha B)$

$\alpha(A+B) = \alpha A + \alpha B$

$(\alpha+\beta)A = \alpha A + \beta A$

If A is invertible and $\alpha \neq 0$, then αA is invertible, and $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

Null Matrix: $a_{ij} = 0 \ \forall i, j$, write 0 for null matrix.

Linear Combinations

$\underset{\sim}{x}^{(1)}, \ldots, \underset{\sim}{x}^{(k)}$ $\quad k$ n-vectors

$b_1 \underset{\sim}{x}^{(1)} + b_2 \underset{\sim}{x}^{(2)} + \cdots + b_k \underset{\sim}{x}^{(k)}$ called a linear combination (where $b_1, \ldots, b_k$ scalars).

Let A be $m \times n$, $\underset{\sim}{a}_j$ = m-vector which is the j^{th} column of A.

For any n-vector $\underset{\sim}{x}$, $\quad A\underset{\sim}{x} = x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2 + \cdots + x_n \underset{\sim}{a}_n$

is a linear combination of the columns of A.

Let $\underset{\sim}{i}_j$ = j^{th} column of I_n = $\begin{array}{r} \\ \\ \text{position } j \rightarrow \\ \\ \\ \end{array}\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ called j^{th} unit vector

Then $\underset{\sim}{a}_j = A \underset{\sim}{i}_j$.

For any n-vector $\underset{\sim}{b}$, $\quad \underset{\sim}{b} = b_1 \underset{\sim}{i}_1 + \cdots + b_n \underset{\sim}{i}_n$

Existence and Uniqueness of Solutions to $A\underset{\sim}{x} = \underset{\sim}{b}$

(A: given $m \times n$ matrix; $\underset{\sim}{b}$: given m-vector; $\underset{\sim}{x}$: unknown n-vector)

(1) If $\underset{\sim}{x} = \underset{\sim}{x}_1$ is a solution of $A\underset{\sim}{x} = \underset{\sim}{b}$, then any other solution $\underset{\sim}{x}_2$ is of the form $\underset{\sim}{x}_2 = \underset{\sim}{x}_1 + \underset{\sim}{y}$ where $\underset{\sim}{y}$ is a solution of the homogeneous system $A\underset{\sim}{y} = 0$.

(2) $A\underset{\sim}{x} = \underset{\sim}{b}$ has at most one solution $\Leftrightarrow$ $A\underset{\sim}{y} = 0$ has only the "trivial solution" $\underset{\sim}{y} = 0$.

(3) Any homogeneous linear system with fewer equations than unknowns has nontrivial solutions

(4) If $A\underset{\sim}{x} = \underset{\sim}{b}$ has a solution for every m-vector $\underset{\sim}{b}$, then $\exists$ $n \times m$ C such that $AC = I_m$.

(5) If $BA = I$, then $A\underset{\sim}{x} = 0$ has only the trivial solution.

(6) If $A\underset{\sim}{x} = \underset{\sim}{b}$ has a solution for every m-vector $\underset{\sim}{b}$, then $m \leq n$

(7) Let A be square $n \times n$. The following are equivalent

  (i) $A\underset{\sim}{x} = 0$ has only $\underset{\sim}{x} = 0$ as solution

  (ii) $A\underset{\sim}{x} = \underset{\sim}{b}$ has a solution for every n-vector $\underset{\sim}{b}$

  (iii) A is invertible.

Linear Independence

$\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n$ n m-vectors

Lin. ind.: $\quad x_1 \underset{\sim}{a}_1 + x_2 \underset{\sim}{a}_2 + \cdots + x_n \underset{\sim}{a}_n = 0 \quad \Rightarrow \quad x_1 = x_2 = \cdots = x_n = 0$

Otherwise, called linearly dependent

Let A be $m \times n$ matrix whose columns are $\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n$

$\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n$ lin. ind $\iff$ $A\underset{\sim}{x} = \underset{\sim}{0}$ has only trivial solution.

Any set of more than m m-vectors is lin. dep.

Basis If every m-vector $\underset{\sim}{b}$ can be written as a lin. comb of $\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n$ (where $\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n$ are lin. ind.), we call $\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n$ a basis the set of all m-vectors)

So $\quad \underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n$ basis $\iff$ $A\underset{\sim}{x} = \underset{\sim}{b}$ has a unique soln. for each $\underset{\sim}{b}$

Thus m must equal n.

Transpose $\quad A \quad m \times n$

$$B = A^T \text{ is } n \times m \qquad b_{ij} = a_{ji}$$

If $A^T = A$, A called symmetric (must be square)

Facts: $\quad (AB)^T = B^T A^T$

$\quad (A^T)^T = A$

$\quad$ If A is invertible $\quad (A^T)^{-1} = (A^{-1})^T$

(sometimes use notation A^{-T} for this matrix)

Scalar product

If $\underset{\sim}{a}, \underset{\sim}{b}$ are real n-vectors, scalar product is

$$\underset{\sim}{b} \cdot \underset{\sim}{a} = \underset{\sim}{b}^T \underset{\sim}{a} = b_1 a_1 + \cdots + b_n a_n$$

Conjugate transpose (or Hermitian transpose)

Complex matrices $\quad (A^H)_{ij} = \overline{a_{ji}} \qquad$ ($\bar{z}$ is conjugate of z)

If A real, $A^H = A^T$

If $A^H = A$, A called Hermitian

Scalar product for complex vectors:

$$\underset{\sim}{b}^H \underset{\sim}{a} = \bar{b}_1 a_1 + \cdots + \bar{b}_n a_n$$

Determinants

Let $\mathcal{S}_n$ denote the symmetric group of degree n, the set of all permutations p of degree n: $p: \{1,2,\cdots,n\} \to \{1,2,\cdots,n\}$ (one-to-one, onto)

There are $n!$ such permutations, so $\mathcal{S}_n$ has $n!$ elements.

Define the sign of p by $\operatorname{sgn}(p) = \begin{cases} 1 & \text{if } p \text{ is an even permutation} \\ -1 & \text{if } p \text{ is an odd permutation} \end{cases}$

Let A be an $n\times n$ matrix. Define the determinant of A by

$$\det(A) = \sum_{p\in\mathcal{S}_n} \operatorname{sgn}(p)\; a_{1,p(1)}\, a_{2,p(2)} \cdots a_{n,p(n)}$$

Examples

$n=1$ $\quad$ [1] even

$\det A = a_{11}$

$n=2$ $\quad$ [1 2] even $\quad$ [2 1] odd

$\det A = a_{11}a_{22} - a_{12}a_{21}$

$n=3$ $\quad$ [1 2 3] even $\quad$ [2 3 1] even $\quad$ [3 1 2] even $\quad$ [3 2 1] odd $\quad$ [1 3 2] odd $\quad$ [2 1 3] odd

$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

$$\begin{array}{ccc:cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ - & - & - \quad + & + & + \end{array}$$

(easy way to remember 3x3 determinant; caution – does not work for 4x4 or bigger)

Theorem If A is upper- or lower-triangular, $\det A = a_{11}a_{22}\cdots a_{nn}$

Proof The only term in the sum which can be nonzero is when p is the identity permutation.

Theorem If A and B are $n\times n$ matrices, $\det(AB) = (\det A)(\det B)$

Theorem A is invertible iff $\det A \neq 0$

Computing Determinants

Perform PLU factorization using Gaussian elimination with pivoting

$$\det A = (\det P)(\det L)(\det U)$$

$\det P = \operatorname{sgn}(p)$ where p is the associated permutation

$\det L = 1$ $\qquad \det U = u_{11}u_{22}\cdots u_{nn}$

$$\det A = \operatorname{sgn}(p)\, u_{11}u_{22}\cdots u_{nn}$$

See book for Cramer's Rule, expansion in minors.

<u>Matrix Form of Row Operations</u>

Multiplying an $n \times n$ matrix A by the following $n \times n$ matrices R (i.e. forming RA) has the stated effect on A:

(i)

$$R = \begin{array}{r} \\ \\ i \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \alpha & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \quad \begin{pmatrix}\text{where} \\ \alpha \neq 0\end{pmatrix}$$

multiplies the i^{th} row of A by α

(ii)

$$R = \begin{array}{r} \\ \\ i \rightarrow \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & \alpha & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \\ \begin{array}{ccccc} & \uparrow j & & & \end{array} \end{array} \quad \begin{pmatrix}\text{where} \\ i \neq j\end{pmatrix}$$

adds α times the j^{th} row of A to the i^{th} row of A

(iii)

$$R = \begin{array}{r} \\ \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & 1 \cdots 1 & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \\ \begin{array}{ccccccc} & & \uparrow i & & \uparrow j & & \end{array} \end{array} \quad \begin{pmatrix}\text{where} \\ i \neq j\end{pmatrix}$$

← (same as I except $r_{ii} = r_{jj} = 0$, $r_{ij} = r_{ji} = 1$)

interchanges the i^{th} and j^{th} rows of A

(This is a permutation matrix.)

<u>Exercise</u> Show that each of the matrices R in (i), (ii), (iii) above is invertible, and that their corresponding inverses are

(i) $R^{-1} = \begin{array}{r} i \rightarrow \end{array} \begin{bmatrix} 1 & & \\ & \frac{1}{\alpha} & \\ & & \ddots \; 1 \end{bmatrix}$, (ii) $R^{-1} = \begin{array}{r} i \rightarrow \end{array} \begin{bmatrix} 1 & & \\ -\alpha & \ddots & \\ \uparrow j & & 1 \end{bmatrix}$, (iii) $R^{-1} = R$

<u>Definition</u> Two linear systems $A\underset{\sim}{x} = \underset{\sim}{b}$ and $\hat{A}\underset{\sim}{x} = \hat{\underset{\sim}{b}}$ are <u>equivalent</u> if they have the same solution set, i.e., $(\forall \underset{\sim}{x} \in \mathbb{R}^n)$ $A\underset{\sim}{x} = \underset{\sim}{b}$ iff $\hat{A}\underset{\sim}{x} = \hat{\underset{\sim}{b}}$

<u>Theorem</u> Suppose we do one of the following operations on a given linear system $A\underset{\sim}{x} = \underset{\sim}{b}$:

(i) multiply one equation by a nonzero constant

(ii) add a multiple of one equation to another

or (iii) interchange two equations

Then the new system $\hat{A}\underset{\sim}{x} = \hat{\underset{\sim}{b}}$ is equivalent to the original system $A\underset{\sim}{x} = \underset{\sim}{b}$.

<u>Proof</u> Each of these operations can be performed by multiplying both sides of the equation $A\underset{\sim}{x} = \underset{\sim}{b}$ by an <u>invertible</u> matrix R. So $\hat{A} = RA$ and $\hat{\underset{\sim}{b}} = R\underset{\sim}{b}$.

Thus $A\underset{\sim}{x} = \underset{\sim}{b} \Rightarrow RA\underset{\sim}{x} = R\underset{\sim}{b} \Rightarrow \hat{A}\underset{\sim}{x} = \hat{\underset{\sim}{b}}$

and $\hat{A}\underset{\sim}{x} = \hat{\underset{\sim}{b}} \Rightarrow R^{-1}\hat{A}\underset{\sim}{x} = R^{-1}\hat{\underset{\sim}{b}} \Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$