## Chapter 1

# Introduction

A *differential equation* involves derivatives of an unknown function of one independent variable (say $u(x)$), or partial derivatives of an unknown function of more than one independent variable (say $u(x, y)$, or $u(t, x)$, or $u(t, x, y, z)$ etc.). Differential equations have been used extensively to model many problems in daily life, in fluid and solid mechanics, biology, material sciences, economics, ecology, sports and computer sciences.[1] Examples include the Laplace equation for potentials, the Navier-Stokes equations in fluid dynamics, biharmonic equations for stresses in solid mechanics, and the Maxwell equations in electro-magnetics.

The main part of this textbook is to learn different linear partial differential equations and some techniques to find their solutions. Solutions to differential equations often have physical meanings such as temperature, velocity and acceleration fields, concentration, populations, trajectories of moving objects, stock price etc. With solutions of some differential equations, for an example, we can compute the drag, lift, and resistance force of a flying airplane. We can use solutions of differential equations to *predict* or compute many physical quantities and use the information to *design* or *control* solutions for practical applications. Often, better understanding of differential equations is essential to *improve mathematical models.* Another part of this textbook is about Fourier series and analysis that have practical applications in wave propagation, radio or television broadcasting, and fast computing based on fast Fourier transforms (FFT), and in solving partial differential equations using series solutions.

However, although differential equations have wide applications, not many can be solved exactly in terms of elementary functions such as polynomials, $\log x$, $e^x$, trigonometric functions ($\sin x$, $\cos x$, ...) etc., and their combinations. Even if a differential equation can be solved analytically, considerable effort and sound

[1]There are other models in practice, for example, statistical models.

mathematical theory are often needed, and the closed form of the solution may be too messy to be useful. If the analytic solution of the differential equation is unavailable or too difficult to obtain, or takes some complicated form that is unhelpful to use, we may try to find an approximate solution using two different approaches

- Semi-analytic methods. Sometimes we can use series, integral equations, perturbation techniques, or asymptotic methods to obtain approximate solutions to differential equations.

- Numerical solution methods. The rapid development in modern computers has provided another powerful tool in solving differential equations, called numerical solutions of differential equations. Nowadays, many applications such as weather forecasts, space shuttles lunches, robots, heavily depend on super computer simulations. There are tons of books, software packages, numerical methods, online classes for solving differential equations numerically, which is a developing area of study and research and provides an effective way in solving many problems that were impossible to solve before.

In this book, we mainly adopt the first approach and focus on either analytic solutions or series solutions.

If a differential equation whose solution has only one independent variable, then the differential equation is called an *ordinary differential equation* (ODE). We should have seen many ODE examples before. Below are two simple examples,

$$\frac{dy}{dx} = x; \qquad \frac{dy}{dx} = y.$$

The solutions to the above ODEs are $y(x) = \frac{x^2}{2} + C$ and $y(x) = Ce^x$, respectively, for arbitrary constant $C$, which means that if we plug the solution into the differential equation, we will get an identity between the left and hand right hand sides of the differential equation.

If a differential equation whose solution has more than one independent variables, then the differential equation is called a *partial differential equation* (PDE). We use the partial derivative symbol $\frac{\partial}{\partial}$ to represent a partial derivative with one particular (independent) variable such as $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ *etc.* Below are some examples.

**Example 1.1.** *Solve the following partial differential equation,*

$$\frac{\partial u}{\partial x} = x \quad or \quad \frac{\partial u}{\partial x}(x,t) = x. \tag{1.1}$$

*Sometimes, we can specify the independent variables in the PDE as in the second expression above to avoid possible confusions.*

**Solution:** In the above PDE, since there is only one derivative with respect to $x$, we can treat the PDE as an ordinary differential equation while regarding the second variable as a parameter (constant) to get $u(x,t) = \frac{x^2}{2} + C(t)$ for any differentiable function $C(t)$. Now we can check that $u(x,t) = \frac{x^2}{2} + C(t)$ is indeed a solution to the PDE. To to so, first we differentiate $u(x,t)$ with respect to $x$ to get $\frac{\partial u}{\partial x} = x + 0$ and plug it into the PDE to have

$$\text{the left hand side} = \frac{\partial u}{\partial x} = x + 0 = \text{the right hand side.}$$

Thus, we have verified that $u(x,t) = \frac{x^2}{2} + C(t)$ is a solution to the PDE for arbitrary differentiable function $C(t)$.

**Example 1.2.** *Check that $u(x,t) = f(x-at)$ is a solution to the partial differential equation,*

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \tag{1.2}$$

*where $a$ is a constant and $f(s)$ is an arbitrary differentiable function.*

**Solution:** First we differentiate $u(x,t)$ with respect to $x$ using the chain rule to get $\frac{\partial u}{\partial x} = f'(x-at)$. Note that since $f(s)$ is a function of one variable, we can use the symbol $f'$. Similarly, we differentiate $u(x,t)$ with respect to $t$ using the chain rule to get $\frac{\partial u}{\partial t} = f'(x-at)(-a)$. We plug the partial derivatives $\frac{\partial u}{\partial x} = f'(x-at)$ and $\frac{\partial u}{\partial t} = f'(x-at)(-a)$ into the PDE to get

$$LHS = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial t} = f'(x-at) + f'(x-at)(-a) = 0 = RHS.$$

Note that $LHS$ and $RHS$ stand for the left hand side and the right hand side, respectively. Thus, we have verified that $u(x,t) = f(x-at)$ is a solution to the partial differential equation.

**Example 1.3.** *Check that $u(x,y) = x^2 + y^2 + C_1x + C_2y + C_3$ is a solution to the partial differential equation,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4, \tag{1.3}$$

*where, $C_1$, $C_2$, and $C_3$ are constants. We should pay attention to the high order differential notations.*

**Solution:** First we differentiate $u(x, y)$ with respect to $x$ and $y$ twice, respectively to have

$$\frac{\partial u}{\partial x} = 2x + C_1, \qquad \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} = 2,$$
$$\frac{\partial u}{\partial y} = 2y + C_2, \qquad \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 2.$$

We plug them into the PDE to get

$$LHS = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 = 4 = RHS.$$

Thus, we have verified that $u(x, y) = x^2 + y^2 + C_1 x + C_2 y + C_3$ is a solution to the partial differential equation.

Note that for ordinary differential equations, the solution can differ by a constant while for partial differential equations, the solution can differ by *functions*. Some examples of ODE/PDE are listed below.

1. Initial value problems (IVP). The canonical form of a first order system is

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0. \tag{1.4}$$

   A higher order ordinary differential equation of one variable can be rewritten as a first order system. For example, a second order ordinary differential equation

$$\begin{aligned} &u''(t) + a(t)u'(t) + b(t)u(t) = f(t), \\ &u(0) = u_0, \quad u'(0) = v_0, \end{aligned} \tag{1.5}$$

   can be converted into a first order system by setting $y_1(t) = u$ and $y_2(t) = u'(t)$ with $y_1(0) = u_0$ and $y_2(0) = v_0$. Note that the two conditions that uniquely determine the solution to the differential equations are all defined at $t = 0$, a distinguished feature of an initial value problem.

2. Boundary value problems (BVP). Below are two examples of an ODE BVP. The first one is one-dimensional,

$$\begin{aligned} &u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \\ &u(0) = u_0, \quad u(1) = u_1. \end{aligned} \tag{1.6}$$

Note that the two conditions above are defined at different points ($x = 0$ and $x = 1$). The second example is a BVP example of a partial differential equation (PDE) in two-dimensions,

$$\begin{aligned} &u_{xx} + u_{yy} = f(x, y), \quad (x, y) \in \Omega, \\ &u(x, y) = u_0(x, y), \quad (x, y) \in \partial\Omega, \end{aligned} \tag{1.7}$$

in a domain $\Omega$ with boundary $\partial\Omega$, where

$$u_x = \frac{\partial u}{\partial x}, \qquad u_{yy} = \frac{\partial^2 u}{\partial y^2} \tag{1.8}$$

and so on for simplicity of notations if there are no confusions occur. The above PDE is linear and classified as *elliptic*. There are two other classifications for linear PDE, namely, *parabolic* and *hyperbolic*, which will be briefly discussed later in this section. The PDE above is called a two dimensional (2D) Poisson equation. If $f(x, y) = 0$, it is a two dimensional Laplace equation.

3. Boundary and initial value problems, *e.g.*,

$$\begin{aligned} &u_t = c^2 u_{xx} + f(x, t), \qquad 0 < x < 1, \\ &u(0, t) = g_1(t), \quad u(1, t) = g_2(t), \qquad \text{BC}, \\ &u(x, 0) = u_0(x), \qquad \text{IC}, \end{aligned} \tag{1.9}$$

where BC stands for boundary condition(s) while IC for initial condition(s). We call $f(x, t)$ a source term. If $f(x, t) = 0$, the PDE is called a one dimensional (1D) heat equation, which is a parabolic PDE. Note that the PDE $u_t = -c^2 u_{xx}$ is called a backward heat equation. A nonzero perturbation at some time instances will result an exponential growth in the solution as $t$ increases. A two dimensional heat equation has the following form

$$u_t = c^2 \left(u_{xx} + u_{yy}\right). \tag{1.10}$$

4. Eigenvalue problems, *e.g.*,

$$\begin{aligned} &u''(x) = \lambda u(x), \\ &u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{1.11}$$

In this example, both the function $u(x)$ (the *eigenfunction*) and the scalar $\lambda$ (the *eigenvalue*) are unknowns.

5. Diffusion and reaction equations, *e.g.*,

$$\frac{\partial u}{\partial t} = \nabla \cdot (\beta \nabla u) + \mathbf{a} \cdot \nabla u + f(u) \tag{1.12}$$

where $\mathbf{a}$ is a constant vector, $\nabla$ is the gradient operator which is the derivative $\nabla u(x) = \frac{du}{dx}$ in 1D, and $\nabla u(x,y) = [\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}]^T$ in 2D, $\nabla \cdot (\beta \nabla u)$ is called a diffusion term, $\mathbf{a} \cdot \nabla u$ is called an advection term, and $f(u)$ is called a reaction term.

6. Wave equations in 1D have the following form

$$u_{tt} = c^2 u_{xx}, \tag{1.13}$$

where $c > 0$ is called the wave speed. The PDE is hyperbolic. 2D wave equations have the general form

$$u_{tt} = c^2 \left( u_{xx} + u_{yy} \right). \tag{1.14}$$

7. Systems of PDEs. The incompressible Navier-Stokes model is an important nonlinear example for modeling incompressible flows:

$$\begin{aligned} &\rho \left( \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \nabla p + \mu \Delta \mathbf{u} + \mathbf{F}, \\ &\nabla \cdot \mathbf{u} = 0. \end{aligned} \tag{1.15}$$

which has three equations in 2D, and four equations in 3D.

In this book, we will consider *linear* PDEs mostly in one dimension (1D) or two dimensions (2D). A 2D linear PDE has the following general form

$$\begin{aligned} &a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} \\ &\qquad + d(x,y)u_x + e(x,y)u_y + g(x,y)u(x,y) = f(x,y), \end{aligned} \tag{1.16}$$

where the coefficients are independent of $u(x,y)$ so the equation is linear in $u$ and its partial derivatives. In the example above, the solution of the 2D linear PDE is sought in some bounded domain $\Omega$. According to the behaviors of the solutions, the PDE (1.16) is classified as the following three categories:

- Elliptic if $b^2 - ac < 0$ for all $(x,y) \in \Omega$,
- Parabolic if $b^2 - ac = 0$ for all $(x,y) \in \Omega$, and
- Hyperbolic if $b^2 - ac > 0$ for all $(x,y) \in \Omega$.

For some well-known PDEs, for examples, heat equations are parabolic; advection and wave equations are hyperbolic; Laplace and Poisson equations are elliptic. Appropriate solution methods typically depend on the equation class.

For a first order system

$$\frac{\partial \mathbf{u}}{\partial t} = A(\mathbf{x})\frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \tag{1.17}$$

the classification is determined from the eigenvalues of the coefficient matrix $A(\mathbf{x})$. The system if hyperbolic if all eigenvalues are real; otherwise it can be elliptic or parabolic.

## 1.1 Further reading

This textbook provides an introduction to differential equations and Fourier analysis. There are many textbooks on this topic. Each textbook has its own characteristics. Some are long and comprehensive; some are more theoretical, and some are problem solving orientated. At the Department of Mathematics, North Carolina State University, the following textbooks have been used by different instructors (an incomplete list).

- Partial Differential Equations with Fourier Series and Boundary Value Problems by Nakhlé H. Asmar [1].
- Applied Partial Differential Equations (Undergraduate Texts in Mathematics) by David J. Logan, [9]
- Introduction to Applied Partial Differential Equations by John M. Davis [2].

Advanced partial differential equations can be found in [3, 6] and many others. We would also recommend students to Schaum's outline series for summaries, applications, solved problems, and practices, [12, 13]. Often the solutions to partial differential equations are complicated especially with series solutions. It is beneficial to use some powerful packages such as Maple [4] or Mathematics for symbolic derivations and visualizations, and Matlab [5] for computations and visualizations. In terms of numerical solution techniques to PDEs, we refer the readers to [7, 8, 10, 14, 15].

## 1.2 Exercises

**E1.1 ODE Review:** Find general solutions or solutions to the following problems.

**(a)**. $y'(x) + y(x) = 1$.

**(b)**. $y'(x) = -\frac{y(x)}{2}$.

**(c)**. $y'(x) = -\frac{x}{2}$, $y(2) = 3$.

**(d)**. $y'(x) - 2y(x) = \sin x$.

**(e)**. $y'(x) + xy(x) = x$, $y(0) = 0$.

**(f)**. $xdy = ydx$.

**E1.2** The hyperbolic sine and cosine functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

**(a)**. Check they are solutions to ODE $y'' - y = 0$.

**(b)**. Express $e^x$ and $e^{-x}$ in terms of hyperbolic sine and cosine functions.

**(c)**. Find the Wronskian of $W(\sinh x, \cosh x)$. Can it be zero?

**E1.3** Find the general solution of the following partial differential equations assuming that the solution is $u(x,t)$.

**(a)**. $\dfrac{\partial u}{\partial x} = 0$.

**(b)**. $\dfrac{\partial u}{\partial t} = 0$.

**(c)**. $\dfrac{\partial u}{\partial t} = f(x)$, where $f(x)$ is a given function.

**(d)**. $\dfrac{\partial^2 u}{\partial x \partial t} = 0$.

**E1.4** Verify that

**(a)**. $u(x,y) = x^2 + y^2$ satisfies the Poisson equation $u_{xx} + u_{yy} = 4$.

**(b)**. $u(x,y) = \log\sqrt{x^2+y^2}$ satisfies the 2D Laplace equation $u_{xx} + u_{yy} = 0$ if $x^2 + y^2 \neq 0$. **Hint:** You can use Maple to verify.

**E1.5** Verify that a solution to the heat equation $u_t = ku_{xx}$ is given by $u(x,t) = \dfrac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}$. It is called the fundamental solution of the 1D heat equation. **Hint:** Maple can be used.

**E1.6** Show that $u(r,\theta) = \log r$ and $u(r,\theta) = r\cos\theta$ are both solutions to the two-dimensional Laplace equation in the polar coordinates,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

## Chapter 2

# First order partial differential equations

One of the simplest first order partial differential equation (PDE) may be the advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad \text{or} \quad u_t + au_x = 0, \tag{2.1}$$

where $a$ is constant at this moment, $t$ and $x$ are independent variables, $u(x,t)$ is the dependent variable that to be solved. In most of applications, $t$ often stands for the time, and $x$ stands for the space, and a is called a wave speed. The PDE is called a one-dimensional, first order, linear, constant coefficient, and homogeneous one. Although there are two independent variables, it is called one-dimensional (1D) advection equation since there is only one space variable $x$. The PDE is classified as a hyperbolic one, and it is also called a one-way wave equation, or a transport equation.

## 2.1 Method of changing variables

There are several ways to find general solutions of an advection partial differential equation. One of them is the method of changing variables. The idea is to change the partial differential equation to an ordinary differential equation (ODE) so that we can use an ODE solution method to solve the problem. A simplest way of changing variables is the following,

$$\begin{cases} \xi = x - at, \\ \eta = t, \end{cases} \qquad \text{or} \qquad \begin{cases} x = \xi + a\eta, \\ t = \eta. \end{cases} \tag{2.2}$$

Under such a transform, we have $u(x,t) = u(\xi + a\eta, \eta)$ denoted as $U(\xi,\eta) = u(\xi + a\eta, \eta)$. Then, we represent the original PDE in terms of the new variables using the chain rule to have

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = -a\frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta},$$

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}.$$

Plug them into the original PDE (2.1), we get

$$\frac{\partial U}{\partial \eta} = 0. \tag{2.3}$$

Integrating both sides above with respect to $\eta$, we get $U(\xi, \eta) = C$. Note that in an ODE, $C$ is an arbitrary constant. But in a PDE, it can be arbitrary differential function of $\xi$, denoted as $f(\xi)$. Thus we get the solution

$$u(x,t) = u(\xi + a\eta, \eta) = U(\xi, \eta) = f(\xi) = f(x - at). \tag{2.4}$$

It is straightforward to check that $u(x, y)$ above is indeed a solution to the PDE (2.1). It is called the *general solution* of the PDE since there is no condition attached to the problem. Note that $u(x,t) = f(x - at) = f(a(x/a - t)) = F(x/a - t)$ and the general solutions can have different expressions that are essentially the same.

**The General Solution of 1D Advection Equation $u_t + au_x = 0$ is $u(x,t) = f(x - at)$ for any differentiable function $f(x)$.**

**Example 2.1.** *The general solution to $2\frac{\partial u}{\partial t} - 3\frac{\partial u}{\partial x} = 0$ is $u(x,t) = F(x + \frac{3}{2}t) = 0$ or $u(x,t) = F(2x + 3t)$ for any differentiable function $F(x)$.*

**Remark 2.1.** *We require $f(x)$ to be differentiable so that $u(x,t)$ satisfies the PDE at every point $(x,t)$. Such a solution is called a classical or strong solution of the PDE. In many applications, however, a function satisfies the PDE almost everywhere but at a few isolated points or lines or surfaces where the solution maybe discontinuous. Such a solution is called a* weak solution.

Note that there are more than one ways of changing variables. In general, we can use

$$\begin{cases} \xi = a_{11}x + a_{12}t, \\ \eta = a_{21}x + a_{22}t, \end{cases} \tag{2.5}$$

where $a_{ij}$'s are parameters of a transformation matrix $A = \{a_{ij}\}$ that satisfies $det(A) \neq 0$. We can choose $a_{ij}$'s so that the PDE in terms of the new variables is simple, like an ODE, so that we can solve it easily. In the discussion above we have $a_{11} = 1$, $a_{12} = -a$, $a_{21} = 0$, and $a_{22} = 1$.

## 2.2 Solution to Cauchy problems

A Cauchy problem is an initial value problem that is defined in the entire space with an initial condition, that is

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad -\infty < x < \infty, \tag{2.6}$$

$$u(x,0) = u_0(x), \tag{2.7}$$

where $u_0(x)$ is a function defined in $(-\infty, \infty)$. Since we know that the general solution is $u(x,t) = f(x - at)$, we have $u(x,0) = f(x) = u_0(x)$. Thus the solution to the Cauchy problem is

$$u(x,t) = u_0(x - at), \tag{2.8}$$

where $u_0(x)$ is called an initial condition. The solution $u(x,t) = u_0(x - at)$ means that the solution at $(x,t)$ is the same as the initial solution at $(x - at, 0)$. When $a > 0$, $x - at < x$, the solution propagates towards right without changing the shape. That is why it is called a one-way wave equation, or advection equation.

**The Solution to a Cauchy Problem of an Advection Equation** $u_t + au_x = 0, \ -\infty < x < \infty, \quad u(x,0) = g(x)$ **is**

$$u(x,t) = g(x - at) \tag{2.9}$$

**for a given differentiable function** $g(x)$**.**

**Example 2.2.** *Let* $a = 2$ *and*

$$u_0(x) = \begin{cases} \sin x & -\pi \leq x \leq \pi, \\ 0 & \textit{otherwise.} \end{cases}$$

The solution to the Cauchy problem is

$$u_0(x,t) = \begin{cases} \sin(x-2t) & -\pi \le x - 2t \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

In Figure 2.1, we plot the solution at $t = 0$ and $t = 2.5$, we can see that the solution is simply shifted to the right.

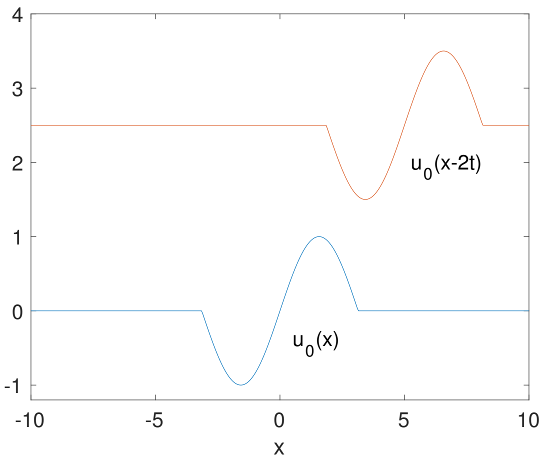



**Figure 2.1.** *Plot of the initial condition $u_0(x)$ and the solution $u(x,t)$ to the advection equation at $t = 2.5$ with the wave speed $a = 2$.*

## 2.3 Method of characteristic for advection equations

A characteristic to a partial differential equation is a set in which the solution to the PDE is a constant (does not change). For a first order PDE of the form $u_t + p(x,t)u_x = f(x,t)$, a characteristic is often a continuous curve $(t(s), x(s))$ with a parameter of $s$, for example, the arc-length of the curve. Let us examine an advection equation $u_t + au_x = 0$ first. Since along the characteristic, the solution $u(x,t) = C$ is a constant, we differentiate the equation on both sides with respect to $t$ to get

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0.$$

Since $u(x,t)$ is the solution to the PDE, we have to have $\frac{dx}{dt} = a$ or $x = at + \bar{C}$. Thus we have $\bar{C} = x - at$. Since $u(x,t)$ is a constant along the line (the characteristic), we have

$$u(x,t) = u(\bar{C}, 0) = u_0(\bar{C}) = u_0(x - at), \tag{2.10}$$

where $u_0(x)$ is the initial condition. Often we can simply write $\bar{C} = C$. Note that in a partial differential equation, an arbitrary constant often corresponds to an arbitrary function, so we have $C = f(x - at) = u(x, t)$. Once again, we get the general solution using a different method. It is important to know that along a curve $\bar{x} - x = a(\bar{t} - t)$, the solution $u(x, t)$ is a constant, which is the basis to determine appropriate boundary conditions for boundary value problems.

## 2.4 Solution of advection equations of boundary value problems

Now consider an initial and boundary value problem of an advection equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad 0 < x < L, \tag{2.11}$$

$$u(x, 0) = u_0(x), \qquad 0 < x < L, \tag{2.12}$$

for a positive constant $L$. We need one or two boundary conditions to make the problem well-posed, that is, the conditions that make the solution exist and unique. Given a point $(x, t)$, $0 < x < L$ and $t > 0$, we can use the method of characteristic to track back the solution to either the initial condition or the boundary condition whichever is the first hit by the characteristic in the domain.

For example, assume that $a > 0$, see the left diagram in Figure 2.2 for an illustration. The line $x = at$ passes through the origin and divide the domain, a strip in the first quadrant, as two parts.

**Solution in the lower right triangle:** In this domain, we should have one of the following, $x < at$ or $x > at$. Which one is it? Usually we can select a point to decide. At the point $x = L/2$, $t = 0$, we have $L/2 > a \cdot 0 = 0$, which means $x > at$. Thus, the domain is characterized as $x > at$. Next, we trace back the solution $u(x, t)$ to the initial condition, not that the boundary condition, why? To do so, we temporarily fix a point $(x, t)$, and write down the characteristic line using $(x, t)$ and the slope $a$ in the $x$-$t$ plane, or $1/a$ in the $t$-$x$ plane,

$$(X - x) = a(T - t), \tag{2.13}$$

where $(x, t)$ is a point that we want to find the solution of $u(x, t)$, and $(X, T)$ is any point on the straight line. If the line intersection the $x$-axis, that is, $T = 0$ for some $X^*$ between zero and $L$, then the solution is determined from the initial condition. By setting $T = 0$, we get $X^* = x - at$. Thus we have

$$u(x, t) = u(X^*, 0) = u_0(X^*) = u_0(x - at), \tag{2.14}$$

which is the same as the solution to the Cauchy problem if $0 < at < x < L$ when $a > 0$.

**Solution in the strip above the right triangle part:** If $0 < x < at$, then the intersection of the line (characteristic) and the $x$-axis is $X^* = x - at < 0$ that is out of the solution domain. The line (characteristic) also intersects the $t$ axis at $X = 0$ for some $T^*$. Thus, we set $X = 0$ to solve for the $T$ to get $-x = a(T^* - t)$, or $T^* = t - x/a$. Thus, the solution is from a boundary condition, say $g(0,t) = g_l(t)$,

$$u(x,t) = u(0,T^*) = g_l(T^*) = g_l\left(t - \frac{x}{a}\right). \tag{2.15}$$

In summary, for $a > 0$, we need to prescribed a boundary condition at $x = 0$, say, $u(0,t) = g_l(t)$, here $g_l(t)$ means the boundary condition at the left end.

**Solution to an Advection Equation of BVP:**

$$\begin{aligned} u_t + au_x = 0, \quad a > 0, \quad 0 < x < L, \\ u(x,0) = u_0(x), \qquad u(0,t) = g_l(t) \end{aligned} \tag{2.16}$$

**is** $$u(x,t) = \begin{cases} u_0(x - at) & 0 < at < x < L,\ t > 0 \\ g_l\left(t - \dfrac{x}{a}\right) & 0 < x < \min\{at, L\} \textbf{ and } t > 0. \end{cases}$$

**Example 2.3.** *Solve the boundary value problem:*

$$\begin{gathered} 2u_t + 3u_x = 0, \quad 0 < x < 3, \\ u_0(x) = \sin(5\pi x), \quad 0 < x < 3, \qquad u(0,t) = \sin(3t). \end{gathered}$$

First we suggest to write the PDE in the standard form $u_t + \frac{3}{2}u_x = 0$. From the formula above, we get the following solution to the BVP,

$$u_(x,t) = \begin{cases} \sin\left(5\pi(x - \dfrac{3t}{2})\right) & 0 < \dfrac{3t}{2} < x < 3,\ t > 0, \\ \sin\left(3(t - \dfrac{2x}{3})\right) & 0 < x < \min\left\{\dfrac{3t}{2}, 3\right\} \text{ and } t > 0. \end{cases}$$

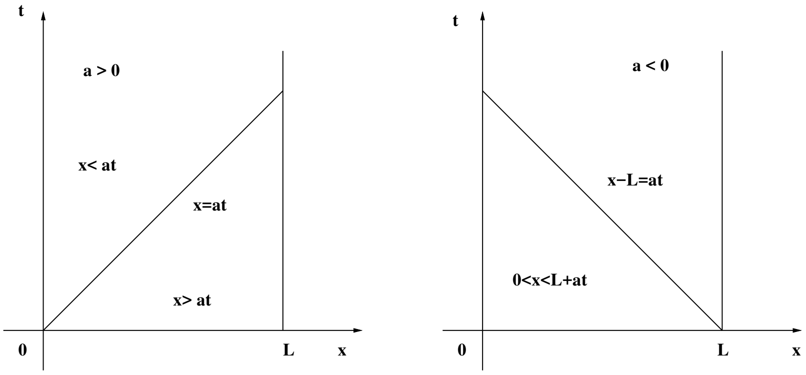



**Figure 2.2.** *Diagrams of the regions where the solution of an advection is determined either by an initial or a boundary condition. The left diagram is for* $a > 0$ *while the right is for* $a < 0$.

## 2.5 Boundary value problems of advection equation with $a < 0$ *

With similar discussions, we know that the initial and boundary value problem,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad 0 < x < L, \tag{2.17}$$

$$u(x,0) = u_0(x), \qquad 0 < x < L, \tag{2.18}$$

with $a < 0$ requires a boundary condition at $x = L$, say, $u(L,t) = g_r(t)$, which needs to be specified, as illustrated in the right diagram in Figure 2.2.

The line equation $x = at$ will be out of the solution domain, which is useless anymore. We should use a line equation like $x = at + C$ that can cut both the axis $t = 0$ (for initial condition) and the boundary $x = L$. Obviously, the line equation $x = at + L$ passes through $(L, 0)$ and divides the domain in the first quadrant as two parts; in one region we have $0 < x < at + L$; in the other region we have $L + at < x < L$. Once again, we can take a point $(L/2, 0)$ to check. Since $L/2 < L$, the triangle region is described by $x < at + L$.

**Solution in the lower left triangle:** In this domain, we have $0 < x < at+L$. Next, we trace back the solution $u(x,t)$ to the initial condition, not the boundary condition, why? To do so, we write down the characteristic line using a point $(x,t)$ and the slope $a$ in the $x$-$t$ plane, or $1/a$ in the $t$-$x$ plane,

$$(X - x) = a(T - t), \tag{2.19}$$

where $(x,t)$ is a point that we want to find the solution of $u(x,t)$, $(X,T)$ is any point on the straight line. If the line intersection the $x$-axis, that is, $T = 0$ for some $X^*$ between zero and $L$, then the solution is determined from the initial condition. By setting $T = 0$, we get $X^* = x - at$. Thus we have

$$u(x,t) = u(X^*,0) = u_0(X^*) = u_0(x - at), \tag{2.20}$$

which is the same as the solution to the Cauchy problem if $0 < x < at + L$.

**Solution in the strip above the right triangle:** If $at + L < x < L$, the intersection of the line (characteristic) and the $x$-axis is $X^* = x - at > L$ or $X^* > at + L > L$ that is out of the solution domain. The line (characteristic) also intersects the line $X = L$ for some $T^*$. Thus, we set $X = L$ to solve for the $T$ to get $L - x = a(T^* - t)$, or $T^* = t + (L - x)/a$ and the solution is from the boundary condition

$$u(x,t) = u\left(L, t + \frac{L-x}{a}\right) = g_r\left(t + \frac{L-x}{a}\right) \quad \text{if } L + at < x < L.$$

In summary, the solution when $a < 0$ is

$$u_0(x,t) = \begin{cases} u_0(x - at) & 0 < x < L + at;\ t > 0, \\ g_r\left(t + \dfrac{L-x}{a}\right) & \max\{0,\ L + at\} < x < L,\ t > 0. \end{cases} \tag{2.21}$$

**Example 2.4.** *Given the boundary value problem below*

$$\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} = 0, \qquad 0 < x < L,$$
$$u(x,0) = \sin(x), \qquad 0 < x < L.$$

*Assume that we know an appropriate boundary condition is* $t\cos t$, *where should it be prescribed,* $x = 0$ *or* $x = L$*? Solve the problem as well.*

**Solution:** In this example $a = -2$, so the slope of characteristics is negative. The line passing through $x = 0$ and $t = 0$ is $x + 2t = 0$ that is out of the solution domain. The line passing through $x = L$ and $t = 0$ with slope $-2$ is $x + 2t = C$. Plugging in $x = L$ and $t = 0$, we get $C = L$. The line $x+2t = L$ divides the solution domain in two regions. The solution in the region bounded by $x = 0$, $t = 0$, and $x + 2t = L$, $0 < x < L$, can trace back to the initial condition.

In another region, the solution can be traced back to the boundary condition at $x = L$. The line equation that passes through $(x,t)$ with slope $-2$ can be written as $(X - x) + 2(T - t) = 0$. Let the intersection of the line with $x = L$ be $(L, t^*)$.

Plug them into the line equation to get $(L-x)+2(t^*-t)=0$ and solve for $t^*$ to get $t^*=t-(L-x)/2$. Thus, the boundary condition should be described at $x=L$ as $u(L,t)=t\cos t$. The solution is

$$u_0(x,t)=\begin{cases}\sin(x+2t) & 0<x<L+2t;\ t>0,\\ \left(t-\dfrac{L-x}{2}\right)\cos\left(t-\dfrac{L-x}{2}\right) & \max\{0,\ L-2t\}<x<L,\ t>0.\end{cases}$$

## 2.6 Method of characteristics for general linear first order PDEs

Consider a general linear and homogeneous first order PDE

$$\frac{\partial u}{\partial t}+p(x,t)\frac{\partial u}{\partial x}=0. \tag{2.22}$$

Using the method of characteristics, we set $\frac{dx}{dt}=p(x,t)$. If we can solve this ODE to get $x-g(t)=C$. Then the general solution to the original problem is $u(x,t)=f(x-g(t))$ for any differentiable function $f(x)$.

**Proof:** If $u(x,t)=f(x-g(t))$ and $\frac{dx}{dt}=-g'(t)=p(x,t)$, then we have $\frac{\partial u}{\partial t}=f'g'(t)=-f'p(x,t)$ and $\frac{\partial u}{\partial x}=f'$. Thus we have $\frac{\partial u}{\partial t}+p(x,t)\frac{\partial u}{\partial x}=f'(-p)+pf'=0$.

**General Solution to an Advection Equation with a Variable Coefficient $\frac{\partial u}{\partial t}+p(x,t)\frac{\partial u}{\partial x}=0$. Use one of two below.**

$$\frac{dx}{dt}=p(x,t),\ x=\int p\,dt+C=f(x,t)+C\Longrightarrow u(x,t)=G(x-f(x,t)).$$

or $$\frac{dt}{dx}=\frac{1}{p(x,t)},\ t=\int\frac{1}{p}dx+C=r(x,t)+C\Longrightarrow u(x,t)=G(t-r(x,t)).$$

**Example 2.5.** *Find the general solution to*

$$\frac{\partial u}{\partial t}+x^2\frac{\partial u}{\partial x}=0.$$

*Find also the solution to the Cauchy problem if* $u(x,0)=\sin x$.

**Solution:** We set $\frac{dx}{dt}=p(x,t)=x^2$ or $\frac{dx}{x^2}=dt$. We get $-\frac{1}{x}=t+C$ or $C=t+\frac{1}{x}$. The general solution is $u(x,t)=f(t+\frac{1}{x})$.

Since we have $u(x,0) = f(-1/x) = \sin x$. Let $y = 1/x$ we get $f(y) = \sin y$. The solution to the Cauchy problem is $u(x,t) = \sin \frac{1}{t+1/x} = \sin \frac{x}{tx+1}$. We verified that the solution satisfies the PDE using the Maple.

**Example 2.6.** *Find the general solution to*

$$\frac{1}{t^2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

*Find also the solution to the Cauchy problem if* $u(x,0) = \sin x$.

**Solution:** We set $\frac{dx}{dt} = p(x,t) = t^2$ or $x = t^3/3 + C$. We get $C = x - t^3/3$. The general solution is $u(x,t) = f(x - \frac{t^3}{3})$.

Since we have $u(x,0) = f(x) = \sin x$. The solution to the Cauchy problem is $u(x,t) = \sin\left(x - \frac{t^3}{3}\right)$, which satisfies the PDE as verified by the Maple.

## 2.7 Solution to first order linear non-homogeneous PDEs with constant coefficients

Using the method of changing variables, we can transform a first order linear non-homogeneous PDEs with constant coefficients

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + bu = f(x,t) \tag{2.23}$$

to an ODE. Thus we can solve the ODE to get the general solution to the PDE. We use the same new variables

$$\begin{cases} \xi = x - at, \\ \eta = t, \end{cases} \qquad \text{or} \qquad \begin{cases} x = \xi + a\eta, \\ t = \eta. \end{cases} \tag{2.24}$$

Under such a transform, we have $u(x,t) = u(\xi + a\eta, \eta)$. We denote $U(\xi,\eta) = u(\xi + a\eta, \eta)$. Then using the chain rule, we can get

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = -a\frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta},$$
$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \xi}.$$

Plug them into the original PDE (2.23), we would get

$$\frac{\partial U}{\partial \eta} + bU = f(\xi + a\eta, \eta) = F(\xi,\eta). \tag{2.25}$$

The equation above is actually ordinary differential equation with respect to $\eta$ (treating $\xi$ as a constant). If we can solve the ODE above, we can get the general solution to the original PDE.

**Example 2.7.** *Find the general solution to*

$$\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial x} - u = t.$$

**Solution:** With the changing variable $\xi = x - 2t$, $\eta = t$, the PDE becomes

$$\frac{\partial U}{\partial \eta} - U = \eta.$$

It is a non-homogeneous ODE and the solution can be expressed as

$$U = U_h + U_p$$

in which $U_h$ is the homogeneous solution to $\frac{\partial U}{\partial \eta} - U = 0$ and $U_p$ is a particularly solution to the ODE. It is easy to get $U_h(\xi, \eta) = g(\xi)e^{\eta}$. From the ODE technique, we can set

$$U_p = A\eta + B$$

for two constants $A$ and $B$. Plug this into the ODE and matching terms on both sides, we get $A = -1$, $B = -1$. Thus the solution in the new variables is

$$U_h(\xi, \eta) = g(\xi)e^{\eta} - \eta - 1.$$

Thus, the general solution to the PDE then is

$$u(x, t) = g(x - 2t)e^{t} - t - 1.$$

**Solution to First Order non-Homogeneous PDE with Constant Coefficients $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + bu = f(x, t)$.**

$\xi = x - at,\ \eta = t, \quad \Longrightarrow \quad \dfrac{\partial U}{\partial \eta} + bU = F(\xi, \eta),$ Assume the solution is

$U(\xi, \eta)$, then the original solution is $u(x, t) = U(x - at, t)$.

**Example 2.8.** *Find the general solution to*

$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial u}{\partial x} + 4u = xt.$$

**Solution:** With the changing variable $\xi = x + \frac{1}{2}t$, $\eta = t$, the PDE becomes

$$\frac{\partial U}{\partial \eta} + 4U = \left(\xi - \frac{1}{2}\eta\right)\eta.$$

It is a non-homogeneous ODE and the solution can be expressed as

$$U = U_h + U_p$$

in which $U_h$ is the homogeneous solution to $\frac{\partial U}{\partial \eta} + 4U = 0$ and $U_p$ is a particularly solution to the ODE. It is easy to get $U_h(\xi,\eta) = g(\xi)e^{-4\eta}$. From the ODE technique, we can set

$$U_p = A\eta^2 + B\eta + C$$

where $A$, $B$, and $C$ are constants. Plug this into the ODE and matching terms on both sides, we get $A = -1/8$, $B = \frac{\xi}{4} + \frac{1}{16}$, $C = -\frac{\xi}{16} - \frac{1}{64}$. Thus the solution in the new variables is

$$U_h(\xi,\eta) = g(\xi)e^{-4\eta} - \frac{\eta^2}{2} + \left(\frac{\xi}{4} + \frac{1}{16}\right)\eta - \frac{\xi}{16} - \frac{1}{64}.$$

Thus the general solution to the PDE then is

$$u(x,t) = g\left(x + \frac{t}{2}\right)e^{-4t} - \frac{t^2}{2} + \left(\frac{x+t/2}{4} + \frac{1}{16}\right)t - \frac{x+t/2}{16} - \frac{1}{64}.$$

## 2.8 Exercises

**E2.1** Classify the following PDE as much as you can (linear, quasi-linear, or non-linear; order; constant or variable coefficient(s); homogeneous or not; dimension(s); type: hyperbolic, elliptic, parabolic; physical meanings: heat, wave, potential ) as much as you can. Also give physical backgrounds if you can.

**(a)**.

$$Au_t = B(u_{xx} + u_{yy}) + f(x,y), \qquad \text{consider } A \neq 0 \text{ and } A = 0.$$

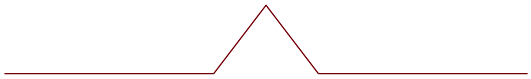
**(b)**.

$$u_{tt} = B(u_{xx} + u_{yy}) + f(x,y)$$

**(c)**.

$$u_t + au_x = Bu_{xx} + f(u).$$

In the expressions above, $A$, $B$, $a$, and $\mu$ are constants.



**E2.2** Given $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0.$

**(a)**. Find the general solution.

**(b)**. Find the solution to the Cauchy problem given $u(x, 0) = 2e^{-2x^2}, \quad -\infty < x < \infty,\ t > 0.$

**(c)**. Sketch of the solution at $t = 3$ if the initial condition $u(x, 0)$ is given

$$u(x,0) = \begin{cases} 1 - |x| & -1 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

see the top plot which is called a hat function. Mark the height of the solution at $t = 3$.

**E2.3** Derive the general solution of the given equation

$$(a), \quad 2u_t + 3u_x = 0; \qquad (b), \quad au_t + bu_x = u, \quad a^2 + b^2 \neq 0.$$

Solve the Cauchy problem with $u(x, 0) = \sin x$, and $u(x, 0) = e^{-x^2}$.

**E2.4** Solve the given partial differential equations below by the method of characteristics. Check you your answer by plugging it back into the equation.

**(a)**.

$$u_t + \sin t u_x = 0.$$

**(b)**.

$$e^{x^2} u_x + xu_y = 0.$$

**E2.5** Find the solution to the following transport equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x} = 0, \qquad 0 < x < 1, \quad t > 0, \\ &u(x, 0) = e^{-x}, \qquad 0 < x < 1, \\ &u(0, t) = t^2, \qquad 0 < t. \end{aligned}$$

**E2.6** Find the solution to the following transport equation * :

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial u}{\partial x} = 0, \qquad 0 < x < 1, \quad t > 0,\\
&u(x,0) = e^{-x}, \qquad 0 < x < 1,\\
&u(1,t) = t^2, \qquad 0 < t.
\end{aligned}$$

**E2.7** Solve the following PDE with $u(x,0) = f(x)$.

**(a)**. $u_t + au_x = e^{2x}$.

**(b)**. $u_t + xu_x = 0$.

**(c)**. $u_t + tu_x = 0$.

**(d)**. $u_t + 3u_x = u + xt$.

**E2.8** Simulate the solution (make plots or movies) using Maple or Matlab for the advection equation $u_t + au_x = 0$ of the Cauchy problem ($-\infty < x < \infty$) with the following initial conditions:

**(a)**.

$$u_0(x) = \begin{cases} \cos(2x) & -2\pi \le x \le 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

**(b)**.

$$u_0(x) = \begin{cases} 1 - |x| & -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 3

# Solution to one dimensional wave equations

A one dimensional (1D) wave equation has the following form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

where $c > 0$ is the wave speed in physics. The partial differential equations is a second order, linear, constant coefficient, homogeneous one. According to the criterion defined on page 8, the PDE is classified as hyperbolic. We first to derive the general solution for which no constraints are imposed.

We can use the method of changing variables to simplify the PDE by setting

$$\begin{cases} \xi = x - ct, \\ \eta = x + ct, \end{cases} \qquad \text{or} \qquad \begin{cases} x = \dfrac{\xi + \eta}{2}, \\ t = \dfrac{\eta - \xi}{2c}. \end{cases} \tag{3.2}$$

Under such a transform, we have $u(x,t) = u\left(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2c}\right) = U(\xi,\eta)$. Then using the chain rule, we can get

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial U}{\partial \xi} + c\frac{\partial U}{\partial \eta},$$
$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}.$$

Differentiating the first expressions above with respect to $t$ again, we get

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= (-c)\frac{\partial^2 U}{\partial \xi^2}\frac{\partial \xi}{\partial t} + (-c)\frac{\partial^2 U}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial t} + c\frac{\partial^2 U}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial t} + c\frac{\partial^2 U}{\partial \eta^2}\frac{\partial \eta}{\partial t} \\ &= c^2\frac{\partial^2 U}{\partial \xi^2} - 2c^2\frac{\partial^2 U}{\partial \xi \partial \eta} + c^2\frac{\partial^2 U}{\partial \eta^2}, \end{aligned} \tag{3.3}$$

assuming both $\frac{\partial^2 U}{\partial\xi\partial\eta}$ and $\frac{\partial^2 U}{\partial\eta\partial\xi}$ are continuous so that they are the same. Similarly we can derive

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 U}{\partial \xi^2} + 2\frac{\partial^2 U}{\partial\xi\partial\eta} + \frac{\partial^2 U}{\partial \eta^2}. \tag{3.4}$$

Plugging (3.3) and (3.4) into the original PDE, we obtain

$$4c^2 \frac{\partial^2 U}{\partial\xi\partial\eta^2} = 0 \qquad \text{or} \qquad \frac{\partial^2 U}{\partial\xi\partial\eta} = 0$$

after some manipulations and using the fact that $c \neq 0$. We integrate $\frac{\partial^2 U}{\partial\xi\partial\eta} = 0$ with respect to $\eta$ to get $\frac{\partial U}{\partial \xi} = f(\xi)$; and integrate it with respect to $\xi$ to further have

$$U(\xi, \eta) = \int f(\xi) d\xi + G(\eta) = F(\xi) + G(\eta)$$

since $\int f(\xi) d\xi$ is still a function of $\xi$. Finally, we substitute the new variables back to the original ones to get the general solution

$$u(x,t) = u\left(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2c}\right) = U(\xi,\eta) = F(x-ct) + G(x+ct) \tag{3.5}$$

for any twice one dimensional differentiable functions $F(x)$ and $G(x)$.

**The General Solution of 1D Wave Equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is**

$$u(x,t) = F(x-ct) + G(x+ct)$$

**for any differentiable function $F(x)$ and $G(x)$.**

**Example 3.1.** *The general solution to* $2\frac{\partial^2 u}{\partial t^2} - 3\frac{\partial^2 u}{\partial x^2} = 0$ *is*

$$u(x,t) = F\left(x + \sqrt{\frac{3}{2}}t\right) + G\left(x - \sqrt{\frac{3}{2}}t\right)$$

*for any differentiable function* $F(x)$ *and* $G(x)$.

## 3.1 Solution to Cauchy problems of 1D wave equations

A Cauchy problem (an initial value problem) of a one-dimensional (1D) wave equation has the following form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \tag{3.6}$$

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad -\infty < x < \infty, \tag{3.7}$$

where $f(x)$ and $g(x)$ are given initial conditions. The solution to the Cauchy problem can be represented by the D'Alembert's formula

$$u(x,t) = \frac{1}{2}\Big( f(x-at) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-at}^{x+ct} g(s)ds. \tag{3.8}$$

**Proof:** First we check the initial conditions. We have

$$u(x,0) = \frac{1}{2}\left(f(x) + f(x)\right) + 0 = f(x)$$

since the integration is zero if the lower and upper limits of the integration are the same. Secondly, we differentiate the equality (3.8) with respect to $t$ first, then let $t = 0$ to get

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\left(f'(x)(-c) + f'(x)c\right) + \frac{1}{2c}\left(cg(x) - g(x)(-c)\right) = g(x).$$

To prove that the function in the D'Alembert's formula satisfies the wave equation, we just need to find $F(x)$ and $G(x)$ in the general solution in terms of $f(x)$ and $g(x)$. From the initial condition, we already have

$$u(x,0) = F(x) + G(x) = f(x). \tag{3.9}$$

Differentiating the general solution $u(x,t) = F(x-ct) + G(x+ct)$ with respect to $t$, we get

$$\frac{\partial u}{\partial t} = F'(x-ct)(-c) + G'(x+ct)c, \tag{3.10}$$

which leads to

$$\frac{\partial u}{\partial t}(x,0) = -F'(x)c + G'(x)c = g(x). \tag{3.11}$$

We further integrate the equality above from 0 to $x$ to get

$$-F(x) + G(x) = \frac{1}{c}\int_0^x g(s)ds + 2A, \tag{3.12}$$

where $A$ is a constant. Note that we use $2A$ for simplicity of derivation. Add (3.12) and $F(x) + G(x) = f(x)$ together we get

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds + A.$$

From (3.9) and the above identity, we also arrive at

$$F(x) = f(x) - G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds - A.$$

Plug $F(x)$ and $G(x)$ above into the general solution, we get

$$\begin{aligned}
u(x,t) = F(x-ct) + G(x+ct) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)ds - A \\
&+ \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + A \\
&= \frac{1}{2}\left(f(x+ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{0} g(s)ds + \frac{1}{2c}\int_0^{x+ct} g(s)ds \\
&= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.
\end{aligned}$$

**Solution to the Cauchy problem of a 1D Wave Equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad -\infty < x < \infty,$$

**is given by the D'Alembert's formula**

$$u(x,t) = \frac{1}{2}\Big(f(x-at) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-at}^{x+ct} g(s)ds.$$

**Example 3.2.** *Solve the Cauchy problem for the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,$$

$$u(x,0) = \begin{cases} \sin x & \text{if } |x| \le \pi, \\ 0 & \text{otherwise}, \end{cases} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

**Solution:** The solution is simply

$$u(x,t) = \frac{1}{2}\left(f(x-2t) + f(x+2t)\right).$$

If $t$ is large enough, then the non-zero regions of $f(x-2t)$ and $f(x+2t)$ do not overlap. We see clearly a single sine wave in the domain $(-\pi, \pi)$ propagates to the right and left with half the magnitude, see the solution plot in Fig. 4.3 for an illustration. In the literature $f(x-ct)$ is called the right-going wave, while $f(x+ct)$ the left-going wave.

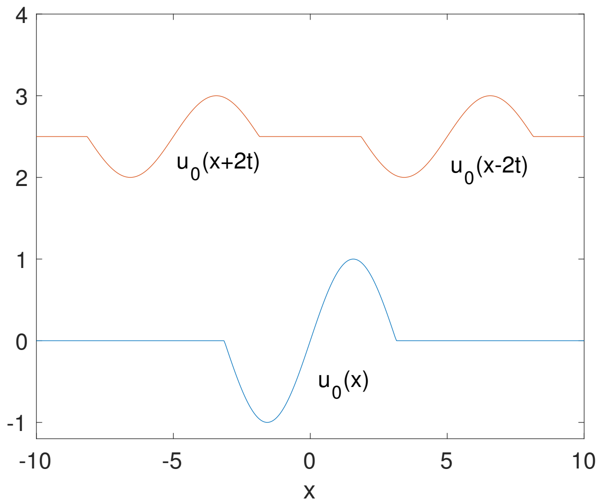



**Figure 3.1.** *Plot of the initial condition $u_0(x)$ and a solution $u(x,t)$ at a later time ($t = 2.5$) to the 1D wave solution with the wave speed $c = 2$.*

**Example 3.3.** *Solve the Cauchy problem for the wave equation*

$$\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty$$

$$u(x,0) = \sin x \qquad \frac{\partial u}{\partial t}(x,0) = xe^{-x^2}.$$

**Solution:** According to the D'Alembert's formula, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\sin(x-\sqrt{2}t) + \sin(x+\sqrt{2}t)\right) + \frac{1}{2\sqrt{2}}\int_{x-\sqrt{2}t}^{x+\sqrt{2}t} se^{-s^2}ds \\ &= \frac{1}{2}\left(\sin(x-\sqrt{2}t) + \sin(x+\sqrt{2}t)\right) + \frac{1}{4\sqrt{2}}\left(e^{-(x-\sqrt{2}t)^2} - e^{-(x+\sqrt{2}t)^2}\right). \end{aligned}$$

**Example 3.4.** *Plot or sketch of the solution of the Cauchy problem for the wave equation below for large $t$.*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty,$$

$$u(x,0) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2(1-x) & \text{if } \frac{1}{2} \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

In Figure 3.2, we show the plot of the solution at time $t = 0$, $t = 0.3$, and $t = 5$. We can see clearly how the one wave split into two with half strength towards left $(x - t)$) and right $(x - t)$). A Matlab movie file is also available (wave_piece.m and fp.m). For this kind of problems when the initial condition is a piecewise continuous function, it is much easy to use a computer to find and plot the solution.

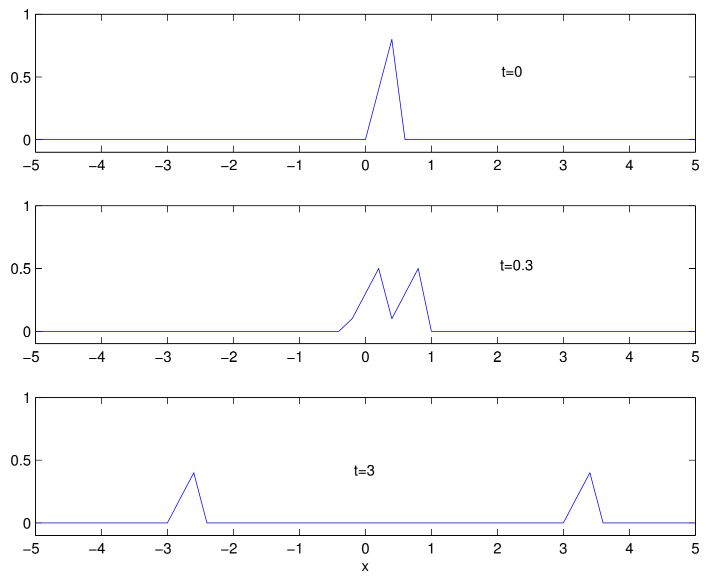



**Figure 3.2.** *Plot of the wave propagation at time $t = 0$, $t = 0.3$, and $t = 5$. We can see clearly how the one wave split into two with half strength towards left $(x - t)$ and right $(x - t)$.*

## 3.2 Normal modes solutions to 1D wave equations of BVPs with special initial conditions

Now consider the boundary value problem of an one dimensional wave equation

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L \\
&u(0,t) = 0, \qquad u(L,t) = 0, \\
&u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L,
\end{aligned}
\tag{3.13}
$$

for a positive constant $L$. An application is an elastic string of a length $L$ with two ends fixed, which corresponds to the homogeneous boundary conditions.

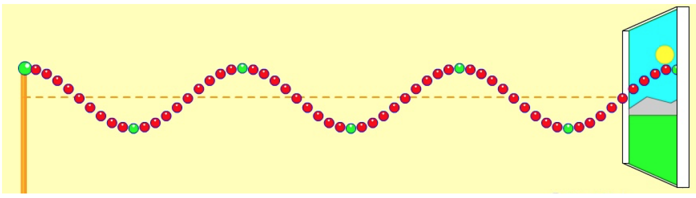


**Figure 3.3.** *A diagram of an elastic string with two ends fixed. The motion can be modeled using a 1D wave equation.*

Since the motion of an elastic string is oscillatory, we would expect the solution is of sort of trigonometric functions of two variables of $x$ and $t$. We can consider $\sin(\alpha x)\cos(\beta t)$, $\sin(\alpha x)\sin(\beta t)$, $\cos(\alpha x)\cos(\beta t)$, $\cos(\alpha x)\sin(\beta t)$, etc. The vanishing boundary condition at $x = 0$ eliminates the $\cos(\beta x)$ option. The solution would look like $\sin(\alpha x)\cos(\beta t)$, $\sin(\alpha x)\sin(\beta t)$. Since the solution is zero at $x = L$, we should have $\sin(\alpha L) = 0$ which means $\alpha L = n\pi$ for $n = 1, 2, \cdots$. Finally the solution should satisfy the PDE, after we differentiate $\sin(\alpha x)\cos(\beta t)$ twice with $x$ and $t$, we will get $\beta = nc\pi/L$.

Thus, a special function

$$
u_n(x,t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L},
\tag{3.14}
$$

for a non-zero integer $n$ can be one of solutions. It is obviously that $u_n(0,t) = u(L,t) = 0$ and $u_n(x,t)$ satisfies the PDE (3.13). Note that $u_n(x,0) = \sin\frac{n\pi x}{L}$ and $\frac{\partial u}{\partial t}(x,0) = 0$. Thus, if $f(x) = \sin\frac{n\pi x}{L}$ and $g(x) = 0$, then $u_n(x,t)$ is the solution to the initial-boundary value problem (3.13). Such a solution is called a normal mode solution to the initial-boundary value problem.

Similarly,

$$\bar{u}_n(x,t) = \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} \tag{3.15}$$

also satisfies the boundary condition and the PDE. Now we have $\bar{u}_n(x,0) = 0$ and $\frac{\partial \bar{u}}{\partial t}(x,0) = \frac{nc\pi}{L}\sin\frac{n\pi x}{L}$. Thus, if $f(x) = 0$ and $g(x) = \frac{nc\pi}{L}\sin\frac{n\pi x}{L}$, then $\bar{u}_n(x,t)$ is a normal mode solution to the initial-boundary value problem (3.13).

**The following normal modes**

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}, \qquad \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L},$$

$$\sum_{n=1}^{N}\sin\frac{n\pi x}{L}\left(A_n\sin\frac{n\pi ct}{L} + B_n\cos\frac{n\pi ct}{L}\right),$$

**satisfy the 1D wave equation and the homogeneous boundary conditions, but NOT arbitrary initial conditions.**

**Example 3.5.** *If $f(x) = \frac{1}{2}\sin\frac{5\pi x}{L}$ and $g(x) = 0$, find the solution to the initial-boundary value problem (3.13).*

***Solution:*** *According to the normal mode solution, the solution is*

$$u(x,t) = \frac{1}{2}\sin\frac{5\pi x}{L}\cos\frac{5\pi ct}{L}.$$

**Example 3.6.** *If $f(x) = 0$ and $g(x) = \frac{1}{2}\sin\frac{5\pi x}{L}$, find the solution to the initial-boundary value problem (3.13).*

***Solution:*** *According to the normal mode solution of the second type, the solution is*

$$u(x,t) = \frac{L}{10\pi c}\sin\frac{5\pi x}{L}\sin\frac{5\pi ct}{L}.$$

**Example 3.7.** *If $f(x) = \sin\frac{5\pi x}{L} - 10\sin\frac{20\pi x}{L}$ and $g(x) = \frac{1}{2}\sin\frac{15\pi x}{L}$, find the solution to the initial-boundary value problem (3.13).*

***Solution:*** *According to the normal mode solutions and the principle of superposition, the solution is*

$$u(x,t) = \sin\frac{5\pi x}{L}\cos\frac{5\pi ct}{L} - 10\sin\frac{20\pi x}{L}\cos\frac{20\pi ct}{L} + \frac{L}{30\pi c}\sin\frac{15\pi x}{L}\sin\frac{15\pi ct}{L}.$$

This is because the PDE is linear, homogeneous, and with homogeneous boundary conditions.

**Challenge:** How about the normal modes of

$$\tilde{u}_n(x,t) = \cos\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}, \qquad \hat{u}_n(x,t) = \cos\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}?$$

From the superposition, we know that the linear combination

$$u_N(x,t) = \sum_{n=1}^{N}\left(a_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\right) \tag{3.16}$$

is the solution to the initial-boundary value problem (3.13) with special initial condition

$$u_N(x,0) = f(x) = \sum_{n=1}^{N} a_n\sin\frac{n\pi x}{L},$$

$$\frac{\partial u_N}{\partial t}(x,0) = g(x) = \sum_{n=1}^{N} b_n\frac{L}{n\pi c}\sin\frac{n\pi x}{L}.$$

What should we do for other general $f(x)$ and $g(x)$? We can use the method of separation variablea and Fourier expansions ($N \to \infty$) that will be discussed later.

## 3.3 Exercises

**E3.1** Let $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$. **(A)**. Find the general solution; **(B)**. Find the solution to the Cauchy problem (given $u(x,0) = f(x)$ and $\dfrac{\partial u}{\partial t}(x,0) = g(x)$).

**(a)**. $c = 1/\pi$, $f(x) = \sin(\pi x)$, $g(x) = 0$.

**(b)**. $c = 1$, $f(x) = \sin(\pi x) + 3\sin(2\pi x)$, $g(x) = \sin(\pi x)$.

**(c)**. Computer project. **(Extra Credit)** $c = 1$, $g(x) = x$,

$$f(x) = \begin{cases} 2\text{ x} & \text{if } 0 \le x \le \frac{1}{2}, \\ 2\text{ (1-x)} & \text{if } \frac{1}{2} \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Plot or sketch the solution at $t = 0.5$ and 1 for all the problems above. Make a movie of the solution between $0 \le t \le 50$.

**E3.2** Let $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < L$. with $u(0,t) = u(L,t) = 0$. Find the solution to the initial and boundary value problem given $u(x,0) = f(x)$ and $\dfrac{\partial u}{\partial t}(x,0) = g(x)$.

**(a)**. $f(x) = \sin \dfrac{2\pi x}{L}, \quad g(x) = 0.$

**(b)**. $f(x) = \dfrac{1}{2} \sin \dfrac{\pi x}{L} + \dfrac{1}{4} \sin \dfrac{3\pi x}{L} + \dfrac{2}{5} \sin \dfrac{7\pi x}{L}, \quad g(x) = 0.$

**(c)**. $f(x) = 0, \quad g(x) = \dfrac{1}{4} \sin \dfrac{3\pi x}{L} - \dfrac{1}{10} \sin \dfrac{6\pi x}{L}.$

**(d)**. $f(x) = \dfrac{1}{4} \sin \dfrac{3\pi x}{L} + \dfrac{1}{10} \sin \dfrac{6\pi x}{L}, \quad g(x) = \dfrac{1}{4} \sin \dfrac{3\pi x}{L} - \dfrac{2}{5} \sin \dfrac{7\pi x}{L}.$

**E3.3** Assume that $c = 2$, $L = 3$, can you solve the 1D wave equation (3.13) with the following $f(x)$ and $g(x)$?

**(a)**. $f(x) = \sin(6\pi x)$, $g(x) = 0$.

**(b)**. $f(x) = 0$, $g(x) = \sin(3\pi x)$.

**(c)**. $f(x) = x \sin x$, $g(x) = 0$.

**(d)**. $f(x) = \sin(6\pi x)$, $g(x) = \sin(3\pi x)$.

**(e)**. $f(x) = \sin(6\pi x) - 7\sin(24\pi x)$, $g(x) = \sin(3\pi x)$.

**E3.4** Given the functions below

$$u_1(x,t) = \sum_{n=1}^{N} \frac{1}{n} \sin \frac{n\pi x}{2} \sin \frac{n\pi\sqrt{3}t}{2}, \qquad u_2(x,t) = \sum_{n=1}^{N} \frac{1}{n} \sin \frac{n\pi x}{2} \cos \frac{n\pi\sqrt{3}t}{2}.$$

**(a)**. Assume that $u(x,0) = u_1(x,0)$ and $\dfrac{\partial u}{\partial t}(x,0) = u_2(x,0)$, and $u(0,t) = u(2,t) = 0$, find the wave equation and the solution with those initial and boundary conditions.

**(b)**. Use a computer software (Maple, Matlab, Mathematica etc.) to plot the functions with $n = 5$, $n = 50$, and $n = 500$. What do you observe?

## Chapter 4

# Orthogonal functions & expansions, and Sturm-Liouville theory

For one-dimensional wave equations with homogeneous boundary conditions, even if there is only one non-zero initial condition that is an elementary function such as $u(x,0) = f(x) = e^x$ or $f(x) = \sum_{i=0}^{N} a_i x^i$ assuming that $u_t(x,0) = 0$, we cannot use a combination of normal mode solutions unless we have infinite number of terms. In fact, we may be able to get a series solution if we can expand $e^x$, for example, as below

$$e^x \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{4.1}$$

We use the symbol $\sim$ to indicate that the above expansion may or may not be identical on both sides. Is this expansion possible? When is this possible? Is this valid in $(0, L)$ or any interval? If the expansion is valid in $(0, L)$, then (4.1) is called an orthogonal functions expansion of $e^x$ in terms of $\left\{\sin \frac{n\pi x}{L}\right\}$. Often we use the pair of braces '{ }' to represent a set. How do we get those orthogonal functions? One of answers is from the Sturm-Liouville eigenvalue theory.

Here we give a glimpse of the method of separation of variables for a 1D wave equation of a boundary value problem,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\\
&u(0,t) = 0, \qquad u(L,t) = 0,\\
&u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L.
\end{aligned}$$

We try a solution that has the special form $u(x,T) = T(t)X(x)$, in which the variables are separated. To satisfy the boundary conditions, we should have $X(0) =$

0 and $X(L) = 0$ since $T(t)$ cannot be zero. Thus, we have $\frac{\partial^2 u}{\partial t^2} = T''(t)X(x)$ and $\frac{\partial^2 u}{\partial x^2} = T(t)X''(x)$. Plugging them into the wave equation we get

$$T''(t)X(x) = c^2 T(t)X''(x) \quad \Longrightarrow \quad \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

In the second expression of above, the left hand side is a function of $t$ while the right hand side is a function of $x$, which is possible only if

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

for some constant $\lambda$. We will see why we use the negative sign in front of $\lambda$ later. Thus, we have

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = X(L) = 0. \tag{4.2}$$

This is called a Sturm-Liouville eigenvalue problem of the boundary value problem since both $\lambda$ and $u(x)$ are unknowns. From ordinary differential equation solution methods and theory we know that the solution is

$$X(x) = C_1 e^{\sqrt{-\lambda}\,x} + C_2 e^{-\sqrt{-\lambda}\,x}.$$

If $\lambda \le 0$, then we have to have $X(x) = 0$ from the boundary condition. $X(x) = 0$ is called a *trivial solution*. Note also that $X(x) = 0$ cannot satisfy the initial condition unless $f(x) = 0$ and $g(x) = 0$, and thus, should be discarded. If $\lambda > 0$, then the solution is

$$X(x) = C_1 \cos(\sqrt{\lambda}\,x) + C_2 \sin(\sqrt{\lambda}\,x).$$

The condition $X(0) = 0$ implies that $C_1 = 0$. Thus, we get $X(x) = C_2 \sin(\sqrt{\lambda} x)$. The condition $X(L) = 0$ implies that $X(L) = \sin(\sqrt{\lambda} L) = 0$, which leads to

$$\sqrt{\lambda}\,L = n\pi, \quad \text{or} \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \cdots,$$

$$\text{and} \quad X_n(x) = \sin\frac{n\pi x}{L}.$$

The functions $\{X_n(x)\} = \{\sin\frac{n\pi x}{L}\}$ above satisfy the ODE and the homogeneous boundary conditions for any natural number $n$, and are called the eigenfunctions of the special Sturm-Liouville eigenvalue problem. Note that, we usually ignore the constant $C_2$ in the expression of $\{X_n(x)\}$ because eigenfunctions can differ by a constant. More important, those eigenfunctions are normal modes as we know. Next, we solve for $T(t)$ using

$$T''(t) + c^2 \lambda_n T(t) = 0, \tag{4.3}$$

where $\lambda_n = (\frac{n\pi}{L})^2$ that have already been found. Therefore, the solution of $T(t)$ is,

$$T_n(t) = b_n \cos(\sqrt{\lambda}\, ct) + b_n^* \sin(\sqrt{\lambda}\, ct) = b_n \cos \frac{n\pi ct}{L} + b_n^* \sin \frac{n\pi ct}{L}.$$

We put $X_n(x)$ and $T_n(t)$ together to get a normal mode solution

$$u_n(x,t) = \sin \frac{n\pi x}{L} \left( b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L} \right), \tag{4.4}$$

which satisfy the PDE, the boundary conditions, but not the initial conditions.

We put all the normal mode solutions together to get a series solution.

$$u(x,t) = \sum_{n=0}^{\infty} \sin \frac{n\pi x}{L} \left( b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L} \right) \tag{4.5}$$

which satisfies the PDE and the boundary conditions. The coefficients of $b_n$ and $b_n^*$ are determined from the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. The question is how and why we can do it. In this chapter, we will present a systematical discussion about how to generate those normal modes and how to obtain $b_n$ and $b_n^*$ from initial conditions.

## 4.1 Orthogonal functions

Orthogonal functions are similar to orthogonal basis in the $R^n$ space in linear algebra. Examples and applications include Fourier series, orthogonal polynomials, approximation theory and methods, and many more. One of notable applications is that we can expand functions in terms of orthogonal functions. Orthogonal functions are also intensively utilized in computational mathematics as approximation tools.

In the $R^n$ space that is composed of all column vectors with $n$ components, the simplest orthogonal basis are $\{\mathbf{e}_i\}_{i=1}^n$. For example, if $n = 3$. we have

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which satisfies

$$(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i^T \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\mathbf{e}_i^T \mathbf{e}_j$ is the sum of the products of corresponding components of $\mathbf{e}_i$ and $\mathbf{e}_j$, which is the Euclidian inner product of the two vectors. For any vector $\mathbf{a} =$

$[a_1, a_2, a_3]^T$, we have $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$. If $\mathbf{b} = \{b_i\}$ is a vector, then the inner product of $a$ and $b$ is defined as $(a, b) = \sum_{i=1}^{3} a_i b_i$.

There are other orthogonal basis in $R^3$, for example,

$$\tilde{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \tilde{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix},$$

also form a normalized orthogonal basis since $\tilde{\mathbf{e}}_i, i = 1, 2, 3$, have a unit length in the Euclidian norm. How do we express any vector in terms of $\{\tilde{\mathbf{e}}_i\}$? It is easy to check the following expressions,

$$\mathbf{a} = \alpha_1\tilde{\mathbf{e}}_1 + \alpha_2\tilde{\mathbf{e}}_2 + \alpha_3\tilde{\mathbf{e}}_3, \qquad \alpha_i = \frac{(\mathbf{a}, \tilde{\mathbf{e}}_i)}{(\tilde{\mathbf{e}}_i, \tilde{\mathbf{e}}_i)} = (\mathbf{a}, \tilde{\mathbf{e}}_i).$$

Similar to the $R^n$ space, we define a functional space which is a set of functions that has operations. All square integrable functions in an interval $(a,\ b)$ form a linear space, called the $L^2(a, b)$ space,

$$L^2(a, b) = \left\{ f(x), \quad \int_a^b |f(x)|^2 dx < \infty \right\}. \tag{4.6}$$

It is a linear space because if $f(x) \in L^2(a, b)$ and $g(x) \in L^2(a, b)$, then their linear combination $w(x) = \alpha f(x) + \beta g(x)$ is also in $L^2(a, b)$ for any constant $\alpha$ and $\beta$. In $L^2(a, b)$ we can define an ***inner product*** similar to that in $R^n$ space as

$$(f, g) = \int_a^b f(x)\bar{g}(x)dx, \tag{4.7}$$

where $\bar{g}(x) = g(x)$ in the real number space and is the conjugate of $g(x)$ is the complex number space. For example, if $f(x) = e^x + i \sin x$, then $\bar{f}(x) = e^x - i \sin x$, where $i = \sqrt{-1}$ is the imaginary unit. We call $f(x)$ and $g(x)$ *orthogonal* (similar to perpendicular in Euclidean geometry) in $L^2(a, b)$ if $(f,\ g) = 0$.

**Example 4.1.** *Let $f(x) = 1$ and $g(x) = \sin x$. Are the two functions are orthogonal in $(0, 2\pi)$? We check the inner product*

$$(f, g) = \int_0^{2\pi} f(x)\bar{g}(x)dx = \int_0^{2\pi} \sin x dx = 0.$$

*Thus, the two functions are orthogonal in the interval $(0,\ 2\pi)$ or any interval of length $2\pi$, but it is not orthogonal in the interval $(0,\ \pi)$.*

The norm of a function $f(x)$ in $L^2(a,b)$ in terms of the $L^2(a,b)$ inner product is defined as

$$\|f\|_2 = \|f\|_{L^2} = \sqrt{(f,f)} = \sqrt{\int_a^b |f(x)|^2 dx}. \tag{4.8}$$

Often the subscript is omitted if there is no confusion occurs, that is, $\|f\| = \|f\|_2$.

**Example 4.2.** *Given an interval $(a,b) = (0,\, 2\pi)$, find the $L^2(a,b)$ norm of $f(x) = 1$ and $g(x) = \cos x$. According to the definition, we calculate,*

$$\|f\|_2 = \sqrt{\int_0^{2\pi} |f(x)|^2 dx} = \sqrt{2\pi};$$

$$\|g\|_2 = \sqrt{\int_0^{2\pi} |g(x)|^2 dx} = \sqrt{\int_0^{2\pi} \cos^2 x\, dx} = \sqrt{\pi}.$$

**Example 4.3.** *We redo the computation but with a different interval $(a,b) = (0,\, \pi)$.*

$$\|f\|_2 = \sqrt{\int_0^{\pi} |f(x)|^2 dx} = \sqrt{\pi};$$

$$\|g\|_2 = \sqrt{\int_0^{\pi} \cos^2 x\, dx} = \sqrt{\frac{\pi}{2}}.$$

Note that there are many different norms, for example,

$$\|f\|_1 = \int_a^b |f(x)| dx, \qquad \|f\|_\infty = \max_{0\le x\le 2\pi} |f(x)|, \tag{4.9}$$

which are not associated with the $L^2(a,b)$ space. In fact, all these norms can be put in a uniform form

$$\|f\|_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p} \tag{4.10}$$

for any $p > 0$. It can be shown that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. Note that only when $p = 2$, the norm is differentiable and it is why the $L^2$ norm has the most useful applications.

There are more than one ways to define an inner product, so is the related norm. An inner product is a special functional[2] of two arguments that satisfies

[2] A function whose arguments are functions.

- $(f,\, g) = \overline{(g,\, f)}$
- $(\alpha f + \beta g,\, h) = \alpha(f,\, g) + \beta(g,\, h)$ for any scalars $\alpha$ and $\beta$.

A norm is also a functional that should satisfy

- $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f(x) = 0$, or $\displaystyle\int_a^b f^2(x)dx = 0$;
- $\|\alpha f\| = |\alpha|\|f\|$ for any scalar $\alpha$;
- $\|f + g\| \leq \|f\| + \|g\|$ which is called the triangle inequality.

All these statements are true in the $R^n$ space. The famous Cauchy-Schwartz inequality is also true in $L^2(a,b)$ space, that is

$$|(f,\, g)| \leq \|f\|_2 \|g\|_2, \qquad \text{or}$$
$$\left| \int_a^b f(x)g(x)dx \right| \leq \sqrt{\int_a^b f(x)^2 dx} \sqrt{\int_a^b g(x)^2 dx}. \tag{4.11}$$

Particularly, if we take $g(x) = 1$, we get

$$\left| \int_a^b f(x)dx \right|^2 \leq (b-a) \int_a^b f(x)^2 dx. \tag{4.12}$$

An example of different inner product is a weighted inner product. Let $w(x)$ be a non-negative function such that $w(x) \geq 0$ and $\int_a^b w(x)dx > 0$. The weighted inner product of $f(x)$ and $g(x)$ is defined as

$$(f,\, g)_w = \int_a^b f(x)\overline{g(x)}w(x)dx. \tag{4.13}$$

The function $f(x)$ and $g(x)$ are orthogonal with respect to $w(x)$ on $(a,b)$ if

$$(f,\, g)_w = \int_a^b f(x)\overline{g(x)}w(x)dx = 0. \tag{4.14}$$

The corresponding norm is then

$$\|f\|_w = \sqrt{(f,\, f)_w} = \sqrt{\int_a^b w(x)|f(x)|^2 dx}. \tag{4.15}$$

We will see an applications of weighted inner products and norms for partial differential equations in polar and spherical coordinates for which $w(r) = r$.

**Example 4.4.** *Find the parameter $a$ such that the two functions $f(x) = 1 + ax$ and $g(x) = \sin x$ are orthogonal with respect to the weight function $w(x) = x$ in $(0, \pi)$.*

*Find the $L^2_w(0,\pi)$ norms of $f(x)$ and $g(x)$. Also find the two normalized orthogonal functions.*

**Solution:** With some calculations or using the Maple, we get $\int_0^\pi (1+ax)\, x \,\sin x dx = \pi(1+a)$. To make the two functions orthogonal with respect to $w(x) = x$, we conclude that $a = -1$. Next, we compute

$$\|f\|_w = \sqrt{\int_0^\pi (1-x)^2\, x\, dx} = \frac{\pi}{\sqrt{12}}\sqrt{3\pi^2 - 8\pi + 6},$$

$$\|g\|_w = \sqrt{\int_0^\pi \sin^2 x \cdot x\, dx} = \frac{\pi}{2}.$$

The two normalized orthogonal functions are

$$\bar{f}(x) = \frac{\sqrt{12}}{\pi\sqrt{3\pi^2 - 8\pi + 6}}\,(1-x), \quad \text{and} \quad \bar{g}(x) = \frac{2}{\pi}\sin x.$$

## 4.2 Function expansions in terms of orthogonal sets

We can represent or approximate a function $f(x)$ in terms of a set of orthogonal functions under some conditions.

**Definition 4.1.** *Let $f_1(x), f_2(x), \cdots, f_n(x), \cdots$ be a set of functions in $L^2(a,b)$, which can also be denoted as $\{f_n(x)\}_{n=1}^{\infty}$. It is called an orthogonal set if $(f_i,\, f_j) = 0$ as long as $i \neq j$ for all $i$ and $j$'s. The orthogonal set is called a* **normalized** *orthogonal set if $\|f_i\| = 1$ for all $i$'s.*

**Example 4.5.** *The following functions,*

$$f_1(x) = \sin x, f_2(x) = \sin 2x, f_3(x) = \sin 3x, \cdots, f_n(x) = \sin nx, \cdots,$$

*or $\{\sin nx\}_{n=1}^{\infty}$ is an orthogonal set in $L^2(-\pi,\pi)$.*

**Proof:** If $m \neq n$, we can verify that

$$\int_{-\pi}^{\pi} \sin nx\, \sin mx dx = \int_{-\pi}^{\pi} -\frac{1}{2}\Big( \cos(m+n)x - \cos(m-n)x\Big)\, dx$$

$$= -\frac{1}{2}\left( \frac{\sin(m+n)x}{m+n}\Bigg|_{-\pi}^{\pi} + \frac{\sin(m-n)x}{m-n}\Bigg|_{-\pi}^{\pi}\right) = 0.$$

Note that if $m = n$, then we have

$$\int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2} dx = \pi. \tag{4.16}$$

Thus, we have $\|f_n\| = \sqrt{\pi}$. The new orthogonal set $\left\{\hat{f}_n(x)\right\} = \{f_n(x)/\sqrt{\pi}\}$ is a normalized orthogonal set.

Note also that the above discussions are true for any interval $(a, a+2\pi)$ of length of $2\pi$ since $\sin nx$, $n \neq 0$, is a periodic function of period $2\pi$.

### 4.2.1 Approximating functions using an orthogonal set

We can expand a function $f(x)$ using an orthogonal set of functions $\{f_n(x)\}$ that has a finite or infinite number of terms literally as

$$f(x) \sim \sum_{n=1}^{N} a_n f_n(x); \qquad f(x) \sim \sum_{n=1}^{\infty} a_n f_n(x). \tag{4.17}$$

While we can always do this, the left and right hand sides of above may not be the same, and that is why we use the '$\sim$' sign. To find out the coefficients $\{a_n\}_{n=1}^{\infty}$, we assume that the equal sign holds and apply the inner product of the above with a function $f_m(x)$ to get

$$\Big(f(x),\ f_m(x)\Big) = \left(\sum_{n=1} a_n f_n(x), f_m(x)\right) = \sum_{n=1} a_n \Big(f_n(x),\ f_m(x)\Big).$$

Since $\{f_n(x)\}_{n=1}$ is an orthogonal set, the right hand side terms are zeros except the $m$-th term, that is

$$\Big(f(x),\ f_m(x)\Big) = a_m \Big(f_m(x),\ f_m(x)\Big) \quad \Longrightarrow \quad a_m = \frac{\big(f(x),\ f_m\big)}{\big(f_m(x),\ f_m(x)\big)}. \tag{4.18}$$

**Expansion of $f(x)$ in terms of an orthogonal set $\{\phi_i(x)\}_{i=1}^{N}$ on an interval $(a,b)$, where $N$ can be $\infty$.**

$$f(x) \sim \sum_{n=1}^{N} a_n \phi_n(x), \qquad a_n = \frac{\big(f(x),\ \phi_n(x)\big)}{\big(\phi_n(x),\ \phi_n(x)\big)} = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi_n^2(x)dx}.$$

**Example 4.6.** *Expand $f(x) = x$ in terms of $\{\sin nx\}$ on $(-\pi, \pi)$.*

We know that $\{\sin nx\}$ is an orthogonal set on $(-\pi, \pi)$. The coefficient $a_n$ is

$$a_n = \frac{\int_{-\pi}^{\pi} x \sin nx dx}{\int_{-\pi}^{\pi} \sin^2 nx dx} = \frac{1}{\pi}\left(-\frac{x\cos nx}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}dx\right)$$
$$= -\frac{2}{\pi}\frac{\cos n\pi}{n}.$$

The expansion then is

$$x \sim 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \cdots = \sum_{n=1}^{\infty}\frac{2(-1)^{n-1}}{n\pi}\sin nx.$$

From the Fourier series theory, we know that the equality sign holds for this case at any $x$ in $(-\pi, \pi)$ but not at two end points $-\pi$ and $\pi$.

**Example 4.7.** *Expand $f(x) = x^2$ in terms of $\{\sin nx\}$ on $(-\pi, \pi)$.*

It is easy to check that $a_n = 0$ for all $n$'s. This is because we have

$$a_n = \frac{\int_{-\pi}^{\pi} x^2 \sin nx dx}{\int_{-\pi}^{\pi} \sin^2 nx dx} = 0.$$

The integrand is an odd function whose integral in a symmetric interval is always 0. Such an expansion is meaningless. This is because the function $f(x) = x^2$ does not share much properties of the orthogonal set $\{\sin nx\}$ on $(-\pi, \pi)$.

We call the orthogonal set $\{\sin nx\}$ on $(-\pi, \pi)$ is incomplete or a subset in the space $L^2(\pi, \pi)$. In Figure 4.1, we show a diagram among function sets in $L^2(a, b)$. The sets $\{\sin nx\}$ and $\{\cos nx\}$ are subsets of $L^2(-\pi, \pi)$. While $\{\sin nx\}$ or $\{\cos nx\}$ is not complete in $L^2(-\pi, \pi)$ meaning that not all the functions in the space can have meaningful expansions by the orthogonal sets, they are complete in some smaller spaces if additional conditions are imposed such as some kind of boundary conditions, even or odd functions *etc.*

It is easy to check that the set $\{\cos nx\}_{n=0}^{\infty}$ is also an orthogonal on $(-\pi, \pi)$. Note that this set includes $f(x) = 1$ when $n = 0$. We can expand $f(x) = x^2$ in terms of $\{\cos nx\}_{n=0}^{\infty}$. But it is meaningless to expand $f(x) = x$ in terms of $\{\cos nx\}_{n=0}^{\infty}$. However, if we put the two orthogonal sets together to form a new set $\{1, \cos nx, \sin nx\}_{n=1}^{\infty}$, then we can show that the new set is another orthogonal set since $\int_{-\pi}^{\pi} \sin mx \cos nx = 0$ for any $m$ and $n$. Any function $f(x)$ in $L^2(-\pi, \pi)$ can be expanded by the orthogonal set,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right), \tag{4.19}$$

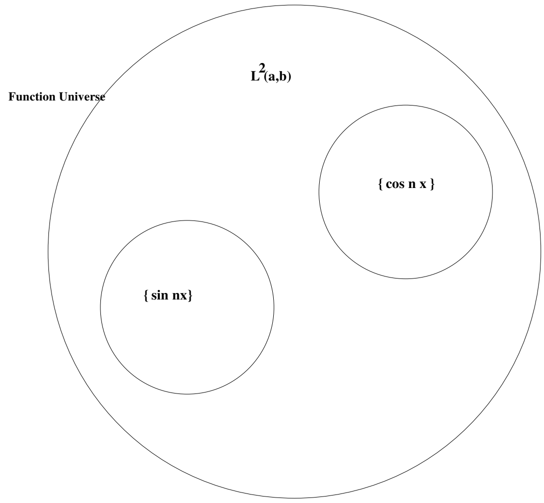



**Figure 4.1.** *A diagram of orthogonal function spaces in $L^2(a,b)$. If we regard all functions as a universe because no one can count them, then $L^2(a,b)$ is a complete subset, called a Hilbert space since an inner product is defined. $L^2(a,b)$ is complete meaning that any Cauchy sequence will converge to a function in $L^2(a,b)$. The orthogonal set $\{\sin nx\}$ and $\{\cos nx\}$ are subsets of $L^2(0,\pi)$.*

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad n = 0, 1, \cdots; \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \qquad n = 1, 2, \cdots \end{aligned} \tag{4.20}$$

This is called a Fourier series of $f(x)$ on $(-\pi,\pi)$. The reason to have a factor $\frac{1}{2}$ for $a_0$ is that we can have a uniform formula for $\{a_n\}_{n=0}^{\infty}$.

## 4.3 Sturm-Liouville eigenvalue problems

Sturm-Liouville (S-L) eigenvalue problems provide a way of generating orthogonal functions that have some special properties. One example is the S-L eigenvalue problem obtained from the method of separation of variables for one-dimensional wave equations $u_{tt} = c^2 u_{xx}$ in the domain $(0,L)$ with homogeneous boundary conditions $u(0,t) = u(L,t) = 0$. The Sturm-Liouville eigenvalue problem would lead to a set of orthogonal functions $\left\{\sin\frac{n\pi x}{L}\right\}$. For any function $f(x) \in L^2(0,L)$ with $f(0) = 0$ and $f(L) = 0$, we can have a meaningful expansion of $f(x)$ in terms of the orthogonal functions $\left\{\sin\frac{n\pi x}{L}\right\}$.

Here we discuss Sturm-Liouville problems that have the following form

$$(p(x)y'(x))' + q(x)y(x) = f(x), \qquad a < x < b \tag{4.21}$$

with boundary conditions (BC) at $x = a$ and $x = b$. Take $x = a$ for example, three types of linear boundary conditions are often used.

**(a).** The solution is prescribed, that is, $y(a) = \alpha$ is known. It is called a Dirichlet BC.

**(b).** The derivative of the solution is prescribed, that is, $y'(a) = \beta$ is known. It is called a Neumann BC. Note that the solution is unknown at $x = a$.

**(c).** The BC is prescribed as $\alpha y(a) + \beta y'(a) = \gamma$ with $\beta \neq 0$. It is called a Robin or mixed BC.

We can write down a uniform form of the three boundary conditions at the two ends as

- $c_1 y(a) + c_2 y'(a) = b_1, \qquad c_1^2 + c_2^2 \neq 0;$
- $d_1 y(b) + d_2 y'(b) = b_1, \qquad d_1^2 + d_2^2 \neq 0.$

The notation $c_1^2 + c_2^2 \neq 0$ means that $c_1$ and $c_2$ cannot be both zero simultaneously. The ordinary differential equation (ODE) is called a self-adjoint ODE. Note that $p(x)y''(x) + w(x)y'(x) + q(x)y(x) = f(x)$ is not a self-adjoint ODE unless it can be transformed to the standard form $(\bar{p}(x)y'(x))' + \bar{q}(x)y(x) = f(x)$.

A Sturm-Liouville problem will have a unique solution if both $p(x)$, $q(x)$, and $f(x)$ are continuous[3], and $p(x) \geq p_0 > 0$ and $q(x) \leq 0$ with suitable boundary conditions, for example, a Dirichlet BC at one of two ends. However, here we are more interested in problems below that have multiple solutions

$$\begin{aligned}
&\Big( p(x)y'(x)\Big)' + \Big( q(x) + \lambda r(x)\Big) y(x) = 0, \qquad a < x < b,\\
&c_1 y(a) + c_2 y'(a) = 0, \qquad c_1^2 + c_2^2 \neq 0,\\
&d_1 y(b) + d_2 y'(b) = 0, \qquad d_1^2 + d_2^2 \neq 0,
\end{aligned} \tag{4.22}$$

with both $y(x)$ and $\lambda$ being unknowns. Such problems are called Sturm-Liouville eigenvalue problems. Note that the ODE and the boundary conditions are all homogeneous and $r(x)$ is a weight function.

[3] These conditions can be lessened in high level mathematics.

Apparently $y(x) = 0$ is a solution, called a *trivial solution.* We can find some $\lambda$ such that the problem has non-trivial solutions. In a Sturm-Liouville eigenvalue problem, we want to find both an eigenvalue $\lambda$, and a corresponding eigenfunction $y_\lambda(x) \neq 0$ that satisfies both of the ODE and the boundary conditions. We call such $((\lambda, y_\lambda(x))$ an eigenpair of the S-L eigenvalue problem.

**Example 4.8.** *Solve the eigenvalue problem*

$$y'' + \lambda y = 0, \quad 0 < x < \pi,$$

$$y(0) = y(\pi) = 0.$$

**Solution:** In this example $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. The roots of the characteristic polynomial of the ODE are $\pm\sqrt{-\lambda}$. If $\lambda < 0$, then the solution is

$$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

Plugging the boundary conditions $y(0) = y(\pi) = 0$ into the above, we get

$$C_1 + C_2 = 0, \qquad C_1 e^{\sqrt{-\lambda}\pi} + C_2 e^{-\sqrt{-\lambda}\pi} = 0.$$

The only solution is $C_1 = 0$ and $C_2 = 0$, which leads to a trivial solution $y(x) = 0$. Similarly if $\lambda = 0$, then $y(x) = C_1 + C_2 x$, which again leads to $y(x) = 0$ using the boundary conditions.

However, if $\lambda > 0$, then the general solution is

$$y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

The boundary condition $y(0) = 0$ leads to $C_1 = 0$. Thus $y(x) = C_2 \sin\sqrt{\lambda}x$. The second boundary condition $y(\pi) = 0$ leads to $C_2 \sin\sqrt{\lambda}\pi = 0$. When does $\sin(x) = 0$? We know that $x = 0$, or $\pi$, or $2\pi$, or $\cdots$, and so on, which leads to $x = n\pi$. Thus, we get

$$\sqrt{\lambda}\pi = n\pi \longrightarrow \lambda = n^2, \quad n = 1, 2, \cdots.$$

Note that $n = 0$ leads to a trivial solution and should be discarded. The solutions to the eigenvalue problem are

$$\lambda_n = n^2, \qquad y_n(x) = \sin nx, \qquad n = 1, 2, \cdots.$$

Usually, we do not include constant $C_2$ term since eigenfunctions can differ by constants. Note also that the eigenfunctions $\{\sin nx\}$ is an orthogonal set in $(0, \pi)$.

## Class practice

Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad 0 < x < 1,$$

$$y(0) = y(1) = 0.$$

The solution is $\lambda_n = (n\pi)^2$ and $y_n(x) = \sin n\pi x$, for $n = 1, 2, \cdots$.

### 4.3.1 Regular and singular Sturm-Liouville eigenvalue problems

Consider again a Sturm-Liouville eigenvalue problem,

$$\begin{aligned} &\big(p(x)y'(x)\big)' + \big(q(x) + \lambda r(x)\big)y(x) = 0, \qquad a < x < b, \\ &c_1 y(a) + c_2 y'(a) = 0, \qquad (c_1)^2 + (c_2)^2 \neq 0, \\ &d_1 y(b) + d_2 y'(b) = 0, \qquad (d_1)^2 + (d_2)^2 \neq 0. \end{aligned} \tag{4.23}$$

Mathematically we require

**1** $p(x)$, $q(x)$ and $r(x)$ are all continuous, or $p, q, r \in C[a, b]$ for short.

**2** $p(x) \geq p_0 > 0$ and $r(x) \geq 0$ for $a \leq x \leq b$,

where $p_0$ is a positive constant. Such a problem is called a *regular Sturm-Liouville* problem. For a regular Sturm-Liouville eigenvalue problem, the eigenfunctions are all continuous and bounded in $(a, b)$. From advanced differential equations theories, we require $p(x)$ is continuous and non-zero in $(a, b)$ so that the ordinary differential equation remains to be a second order ODE. If the conditions, especially, the condition on $p(x)$ is violated, we called the Sturm-Liouville eigenvalue problem, a *singular* problem. Below are some examples:

$$y'' + \lambda y = 0, \quad -1 < x < 1, \quad \text{regular;}$$

$$(xy')' + \lambda y = 0, \quad -1 < x < 1, \quad \text{singular at } x = 0;$$

$$((1 - x^2)y')' + \lambda y = 0, \quad -1 < x < 1, \quad \text{singular at } x = \pm 1.$$

Sometime, we need some effort to re-write a problem to have a standard Sturm-Liouville eigenvalue form to judge whether the problem is regular or singular.

**Example 4.9.** $x^2 y'' + 2xy' + \lambda y = 0$ *can be written as* $(x^2 y')' + 2xy' + \lambda y - 2xy' = 0$, *which is* $(x^2 y')' + \lambda y = 0$. *The eigenvalue problem is a regular in an interval* $(a, b)$

*that does not contain the original, and that the zero is not one of two end points. Otherwise it would be singular.*

**Example 4.10.** *We can divide by* $x^2$ *for* $xy'' - y' + \lambda xy = 0$ *to get* $\frac{1}{x}y'' - \frac{1}{x^2}y' + \lambda y = 0$ *which is* $\left(\frac{1}{x}y'\right)' + \lambda y = 0$*, which is a standard S-L eigenvalue problem. The discussion in the previous example about whether the problem is regular or singular also applies to this S-L eigenvalue problem.*

Below we present an example for a different boundary condition, a Neumann boundary condition at $x = b$.

**Example 4.11.** *Solve the eigenvalue problem*

$$y'' + \lambda y = 0, \quad 0 < x < \pi,$$

$$y(0) = 0, \qquad y'(\pi) = 0.$$

**Solution:** From previous examples, we know that the solution should be $y(x) = C_2 \sin\sqrt{\lambda}x$. Thus the derivative is $y'(x) = C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$. From $y'(\pi) = 0$ we get $y'(\pi) = \cos\sqrt{\lambda}\pi = 0$. Thus, the eigenvalues are

$$\sqrt{\lambda}\pi = \left(\frac{1}{2} + n\right)\pi, \quad n = 0, 1, 2, \cdots, \quad \Longrightarrow \quad \lambda_n = \left(\frac{1}{2} + n\right)^2,$$

$$y_n(x) = \sin\left(\frac{1}{2} + n\right)x.$$

**Question:** Can we take $n = -1, n = -2, \cdots$?

The eigenfunctions $\left\{\sin(\frac{1}{2} + n)x\right\}_{n=0}^{\infty}$ form an orthogonal set that can be used to solve the wave equations $u_{tt} = c^2 u_{xx}$ with the boundary condition $u(0, t) = 0$ and $\frac{\partial u}{\partial x}(\pi, t) = 0$ on the interval $(0, \pi)$.

**Example 4.12.** *Solve the eigenvalue problem with a mixed boundary condition (also called a Robin BC)*

$$y'' + \lambda y = 0, \quad 0 < x < 1,$$

$$y'(0) = 0, \qquad y(1) + y'(1) = 0.$$

**Solution:** From previous discussions, we know that the solution should have the form

$$y(x) = y(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x),$$
$$y'(x) = -\sqrt{\lambda}\, C_1 \sin(\sqrt{\lambda} x) + \sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda} x).$$

From $y'(0) = 0$, we conclude that $C_2 = 0$ since $\sqrt{\lambda} = 0$ implies a trivial solution $y = 0$. From the mixed boundary condition we have

$$C_1 \left(\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}\right) = 0, \quad \text{or} \quad \cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda} = 0, \quad \Longrightarrow \quad \cot\sqrt{\lambda} = \sqrt{\lambda}.$$

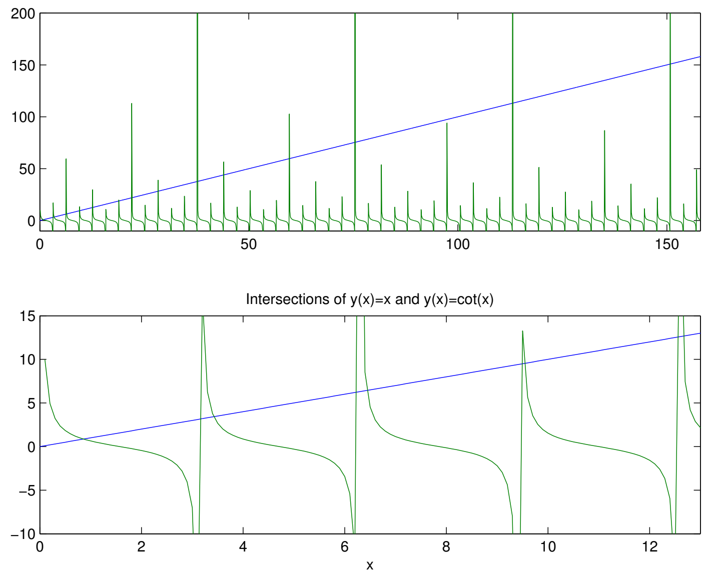



**Figure 4.2.** *Plots of $y = x$ and $y = \cot x$. The intersections are the squares of eigenvalues of $\lambda_n$. Note that round-off errors from the computer and the effect of the singularities of $\cot x$ at $k\pi$, $k = 1, 2, \cdots$, are visible.*

There is no closed form for the eigenvalues, which are zeros of a non-linear equation. But we do know that the squares of the eigenvalues are the intersections of the graphs of $y = x$ and $y = \cot x$ in the half plane $x > 0$. There are infinite number of intersections in the first quadrant at: $\alpha_1 = 0.86\cdots$, $\alpha_2 = 3.43\cdots$, $\alpha_3 = 6.44\cdots$. The eigenvalues are $\lambda_n = \alpha_n^2$, and the eigenfunctions are $y_n(x) = \cos\lambda_n x$. In Figure 4.2, we show two plots of $y = x$ and $y = \cot x$. The intersections are $\alpha_n = \sqrt{\lambda_n}$.

## 4.4 Theory and applications of Sturm-Liouville eigenvalue problems

For a regular Sturm-Liouville eigenvalue problem,

$$\begin{aligned}
&\Big(p(x)y'(x)\Big)' + \Big(q(x)+\lambda r(x)\Big)y(x) = 0, \qquad a<x<b,\\
&c_1 y(a) + c_2 y'(a) = 0, \qquad c_1^2+c_2^2 \neq 0,\\
&d_1 y(b) + d_2 y'(b) = 0, \qquad d_1^2+d_2^2 \neq 0.
\end{aligned} \tag{4.24}$$

Assume that all $p(x), q(x), r(x) \in C(a,b)$ are real functions, $p(x) \geq p_0 > 0$, $r(x) \geq 0$, $\int_a^b r(x) > 0$. Then we have the following theorem.

**Theorem 4.2.**

**1** *There are infinite number of eigenvalues, which are all real numbers. We can arrange the eigenvalues as*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{4.25}$$

*Furthermore, If $q(x) \leq 0$, then all the eigenvalues are positive $\lambda_n > 0$ for $n = 1, 2, \cdots$.*

**2** *The eigenfunctions $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x), \cdots$, form an orthogonal set with respect to the weight function $r(x)$ on $(a,b)$, that is*

$$\int_a^b y_m(x) y_n(x) r(x) dx = 0, \qquad if \quad m \neq n. \tag{4.26}$$

**3** *For any function $u(x) \in L_r^2(a,b)$ that satisfies the same boundary condition, $u(x)$ can be expanded in terms of the orthogonal set $\{y_n(x)\}_{n=1}^{\infty}$, that is,*

$$u(x) = \sum_{n=1}^{\infty} A_n y_n(x) \qquad with \quad A_n = \frac{\displaystyle\int_a^b u(x) y_n(x) r(x) dx}{\displaystyle\int_a^b y_n^2(x) r(x) dx}. \tag{4.27}$$

**Sketch of the proof of the orthogonality:** Let $y_k(x)$ and $y_j(x)$ be two distinct eigenfunctions corresponding to the eigenvalues of $\lambda_k$ and $\lambda_j$, respectively, that is,

$$\left(py_j'\right)' + \left(q + \lambda_j r\right) y_j = 0, \tag{4.28}$$

$$\left(py_k'\right)' + \left(q + \lambda_k r\right) y_k = 0. \tag{4.29}$$

We multiply (4.29) by $y_j(x)$ and multiply (4.28) by $y_k(x)$; and then subtract the two to get

$$y_k\left(py_j'\right)' - y_j\left(py_k'\right)' + (\lambda_j - \lambda_k) r y_j y_k = 0. \tag{4.30}$$

Integrating above from $a$ to $b$ leads to

$$(\lambda_j - \lambda_k)\int_a^b r y_j y_k dx = \int_a^b y_k\left((py_j')' - y_j(py_k')'\right) dx.$$

Applying integration by parts to the right hand side and carrying out some manipulations, we get

$$\begin{aligned}(\lambda_j - \lambda_k)\int_a^b r y_j y_k dx &= p(b)y_j'(b)y_k(b) - p(a)y_j(b)y_k'(b)\\ &\quad - p(a)y_j'(a)y_k(a) + p(a)y_j(a)y_k'(a).\end{aligned}$$

From the boundary condition at $x = a$ we have

$$\begin{bmatrix} y_j(a) & y_j'(a) \\ y_k(a) & y_k'(a) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $c_1^2 + c_2^2 \neq 0$, we must have that the determinant of the 2 x 2 coefficient matrix must be zero, that is, $y_j'(a)y_k(a) - y_j(a)y_k'(a) = 0$. Since $p(a) \neq 0$, we conclude that

$$p(a)y_j'(a)y_k(a) - p(a)y_j(a)y_k'(a) = 0.$$

By the same derivation at $x = b$, we also have

$$p(b)y_j'(b)y_k(b) - p(b)y_j(b)y_k'(b) = 0.$$

Thus, we have $(\lambda_j - \lambda_k)\int_a^b r y_j y_k dx = 0$ and since $\lambda_j \neq \lambda_k$, we conclude that $\int_a^b r y_j y_k dx = 0$. This completes the proof.

## 4.5 Application of the S-L eigenvalue theory and orthogonal expansions

Let us revisit the initial and boundary value problem of one-dimensional wave equations,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \\
&u(0,t) = 0, \qquad u(L,t) = 0, \\
&u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L,
\end{aligned}$$

for general $f(x)$ and $g(x)$. As discussed at the beginning of the chapter, the solution can be expressed as a superposition of normal mode solution,

$$u(x,t) = \sum_{n=0}^{\infty} \sin \frac{n\pi x}{L} \left( b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L} \right) \tag{4.31}$$

that satisfies the PDE and the boundary conditions. The coefficients of $b_n$ and $b_n^*$ are determined from the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Using the orthogonal expansion process, we have

$$\begin{aligned}
&u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad \Longrightarrow \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \\
&\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} \sin \frac{n\pi x}{L} \left( -b_n \frac{cn\pi}{L} \sin \frac{cn\pi t}{L} + b_n^* \frac{cn\pi}{L} \cos \frac{cn\pi t}{L} \right), \\
&\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \sin \frac{cn\pi t}{L} b_n^* \frac{cn\pi}{L} \qquad \Longrightarrow \qquad b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx.
\end{aligned}$$

Thus, the coefficients $b_n$'s are the coefficients of the orthogonal expansions of $u(x,0) = f(x)$ in terms of the eigenfunctions; while the coefficients $b_n^*$ are the coefficients of the orthogonal expansions of $u_t(x,0) = g(x)$ in terms of the eigenfunctions differed by some constants.

**Solution to the 1D wave equation with homogeneous BCs**

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L} \right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \qquad b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx.$$

**Example 4.13.** *Solve the wave equation,*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1,$$

$$u(0,t) = 0, \qquad u(1,t) = 0,$$

$$u(x,0) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

**Solution:** In this example, $c = 1$, $L = 1$, and $g(x) = 0$, we have $b_n^* = 0$ and

$$b_n = 2\int_0^{\frac{1}{2}} f(x) \sin n\pi x dx = 2\int_0^{\frac{1}{2}} \sin n\pi x dx = -\frac{2}{n\pi} \cos n\pi x \Big|_0^{\frac{1}{2}}$$
$$= -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right) = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right).$$

The solution to the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) \sin n\pi x \cos n\pi t.$$

We know that the series is convergent in the interval (0, 1). In Figure 4.3, we show several plots of the partial sums defined as

$$S_n(x) = \sum_{n=1}^{N} \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) \sin n\pi x \cos n\pi t. \tag{4.32}$$

with $N = 1$, $N = 5$, and $N = 75$ at $t = 0$. The series approximates the function $u(x,0)$ well in the interior of continuous regions when $N$ is large enough but oscillates at $x = 0$ as well as at the discontinuity $x = 1/2$, which is called the Gibb's phenomena. In the Maple file, one can use the animation feature to see the evolution of the solution with time $t$. It is interesting to observe how the discontinuity moves.

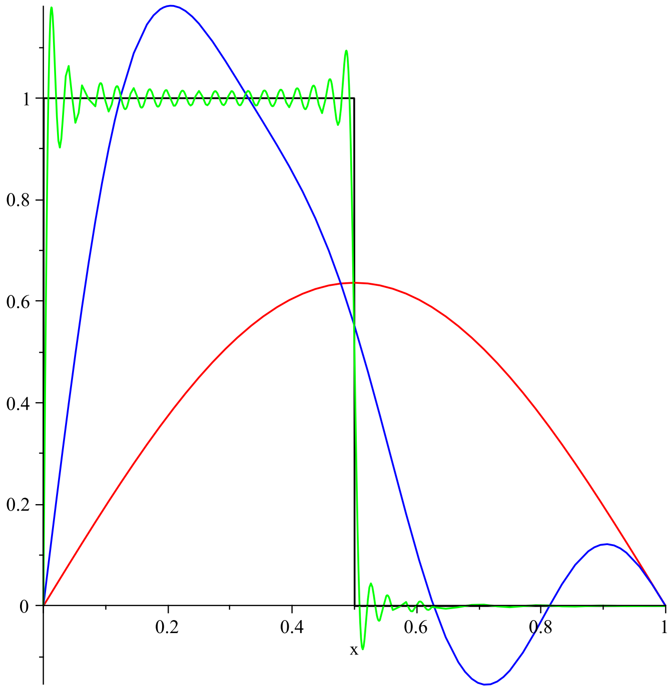



**Figure 4.3.** *Plots of the series approximations (partial sums) of the initial condition to the wave equation.*

**Example 4.14.** *Solve the 1D wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1,$$

$$u(0,t) = 0,$$

$$quad u(1,t) = 0,$$

$$u(x,0) = \sin \pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

**Solution:** For this example, the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are some normal modes and in the expansion forms already. Thus, the solution is a combination of normal mode solutions,

$$u(x,t) = \sin \pi x \cos \pi ct - \frac{1}{2}\sin 2\pi x \cos 2\pi ct + \frac{1}{3}\sin 3\pi x \cos 3\pi ct.$$

## 4.6 Series solution of 1D heat equations of initial and boundary value problems

We use the method of separation of variables to solve,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,$$

$$u(0,t) = 0, \qquad u(L,t) = 0,$$

$$u(0,t) = f(x), \qquad 0 < x < L,$$

for a general $f(x) \in L^2(0,L)$. The PDE is called a one-dimensional heat equation. Note that there is only one initial condition. From the initial and boundary conditions, we should have $f(0) = u(0,0) = 0$ and $f(L) = u(L,0) = 0$. If these two conditions are satisfied, we call the initial and boundary conditions are consistent, which is not always true in some applications. The method of separation of variables includes the following steps.

**Step 1:** Let $u(x,t) = T(t)X(x)$ and we plug its partial derivatives into the original PDE so that we can separate variables. The homogeneous boundary conditions require $X(0) = X(L) = 0$. Differentiating with $u(x,t) = T(t)X(x)$ with $t$ and $x$ respectively, we get

$$\frac{\partial u}{\partial t} = T'(t)X(x); \qquad \frac{\partial u}{\partial x} = T(t)X'(x), \quad \frac{\partial^2 u}{\partial x^2} = T(t)X''(x).$$

The 1D heat equation can be re-written as

$$T'(t)X(x) = c^2 T(t)X''(x) \implies \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \tag{4.33}$$

where $\lambda$ is a constant for given $x$ and $t$. This is because in the last equality, the left hand side is a function of $t$ while the right hand side is a function of $x$, which is possible only both of them are a constant independent of $t$ and $x$. We need to decide which is an eigenvalue problem that we can solve. Since we know the boundary condition for $X(x)$, naturally we should solve

$$\frac{X''(x)}{X(x)} = -\lambda \quad \text{or} \quad X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0 \tag{4.34}$$

first.

**Step 2:** Solve the eigenvalue problem. From the Sturm-Liouville eigenvalue theory, we know that $\lambda > 0$. Thus, the solution is

$$X''(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x.$$

From the boundary condition $X(0) = 0$, we get $C_1 = 0$. From the boundary condition $X(L) = 0$, we have

$$C_2 \sin \sqrt{\lambda} L = 0, \quad \Longrightarrow \quad \sqrt{\lambda} L = n\pi, \quad n = 1, 2, \cdots,$$

since $C_2 \neq 0$ for non-trivial solutions. The eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{L}, \quad n = 1, 2, \cdots.$$

Next, we solve for $T(t)$ using

$$T'(t) + c^2 \lambda_n T(t) = 0, \tag{4.35}$$

with known $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. The solution is (not an eigenvalue problem anymore since we have already known $\lambda_n$)

$$T_n(t) = b_n e^{-c^2 \lambda_n t} = b_n e^{-c^2 (\frac{n\pi}{L})^2 t}.$$

Put $X_n(x)$ and $T_n(t)$ together, we get a normal mode solution

$$u_n(x, t) = b_n e^{-c^2 (\frac{n\pi}{L})^2 t} \sin \frac{n\pi x}{L}, \tag{4.36}$$

which satisfy the PDE, the boundary conditions, but not the initial condition.

**Step 3:** Put all the normal mode solutions together to get the series solution. The coefficients are obtained from the orthogonal expansion of the initial condition.

The solution to the 1D heat equation can be written as

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-c^2 (\frac{n\pi}{L})^2 t} \tag{4.37}$$

which satisfies the PDE and the boundary conditions. The coefficients of $b_n$ are determined from the initial conditions $u(x, 0)$,

$$u(x, 0) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad \Longrightarrow \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

**Series solution to the 1D heat equation is**

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-c^2 (\frac{n\pi}{L})^2 t}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

**Example 4.15.** *Solve the 1D heat equation with homogeneous boundary conditions in the interval* $(0, \pi)$ *and the initial condition* $u(x, 0) = 100$*. What is the limit of* $\lim_{t\to\infty} u(x, t)$*?*

**Solution:** We use the formula above to find the coefficients of the series

$$b_n = \frac{2}{\pi}\int_0^{\pi} 100\sin(nx)dx = \frac{200}{n\pi}\cos(nx)\Big|_0^{\pi} = \frac{200(1-\cos(n\pi)}{n\pi}.$$

Thus, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200\,(1-\cos(n\pi))}{n\pi}\sin(nx)e^{-n^2 t}.$$

Furthermore, we can easily show that $\lim_{t\to\infty} u(x, t) = 0$.

In Figure 4.4, we show plots of several partial sums of the initial condition with $N = 1$, $N = 5$, and $N = 175$. In the middle, the series approximate the function very well when $N$ is large enough but oscillates at two end points, which is called the Gibb's phenomena. However, for heat equations, the oscillations will soon be dampened and the solution becomes smooth with the time. In the Maple file, one can use the animation feature to see the evolution of the solution.

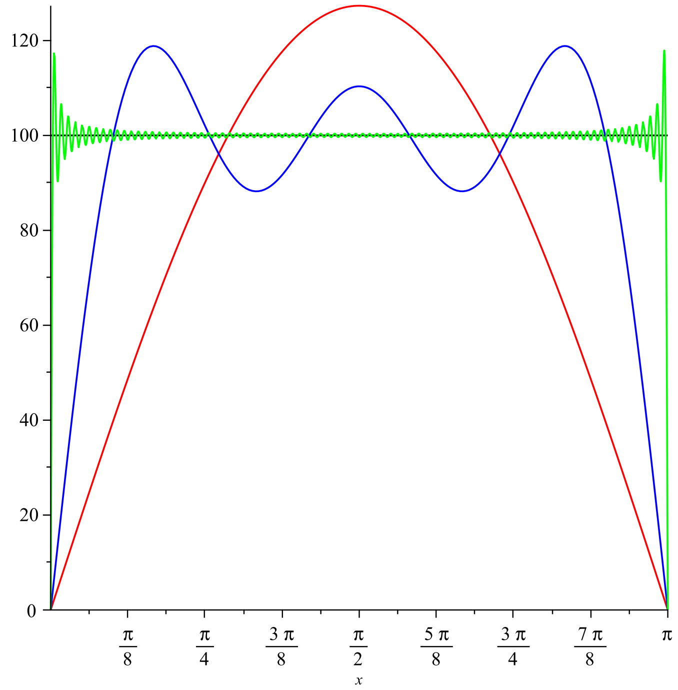



**Figure 4.4.** *Plots of the series approximations to the initial condition* $u_0(x, 0) = 100$.

## 4.7 Exercises

**E4.1** Find all values of $b$ such that the following two functions are orthogonal functions with respect to the weight function $w(x) = e^{-x}$ on the interval $0 < x < \pi$,

$$f_1(x) = \cos(bx), \quad f_2(x) = e^x.$$

**E4.2** Given $\left\{\cos\dfrac{n\pi x}{p}\right\}_{n=0}^{N} = \left\{1, \cos\dfrac{\pi x}{p}, \cos\dfrac{2\pi x}{p}, \cdots, \cos\dfrac{i\pi x}{p}, \cdots, \cos\dfrac{N\pi x}{p}\right\}.$

**(a)**. Show that the set forms an orthogonal set in $(-p, p)$. **Hint:** $\cos\alpha\cos\beta = \dfrac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$.

**(b)**. Find the $L^2$ norm $\|f\|_{L^2} = \sqrt{\int_{-p}^{p} f^2(x)dx}$ of $f(x) = 1$ $(n = 0)$ and $f(x) = \cos\dfrac{n\pi x}{p}$ (other $n$).

**(c)**. Find the orthogonal expansion of $f(x) = |x|$ in terms of the orthogonal set in $(-p, p)$.

**Hint:** $\displaystyle\int x\cos ax dx = \frac{x\,\sin ax}{a} + \frac{\cos ax}{a^2}$.

**E4.3** Determine the constants $a$ and $b$ so that the functions 1, $x$, and $a + bx + x^2$ are orthogonal on $(-1,\, 1)$. Find the orthogonal function expansion of $\sin x$ or $\cos x$ (**choose one**), for example,

$$\sin x \sim \alpha_1 + \alpha_2 x + \alpha_3(a + bx + x^2)$$

after you have found $a$ and $b$. Plot the function *sinx* and the approximation together.

**E4.4** Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be a set of orthogonal functions with respect to a weight function $w(x)$ in an interval $(a, b)$.

**(a)**. What does this mean?

**(b)**. If $f(x) = \displaystyle\sum_{n=1}^{\infty} a_n\phi_n(x)$, then find the integral formula for $a_n$ in terms of the appropriate functions.

**E4.5** Find all the eigenvalues and eigenfunctions of the S-L. problem. Show all the cases and process.

$$y'' + \lambda y = 0, \quad 0 < x < \pi/2;$$

$$y(0) = 0, \quad y'(\pi/2) = 0$$

Also answer the following questions:

**(a)**. Can the eigenvalues of regular S-L be complex numbers?

**(b)**. Are there finite or infinite number of distinct eigenvalues?

**(c)**. If $y_m(x)$ and $y_n(x)$ are two eigenfunctions corresponding to two different eigenvalues $\lambda_m$ and $\lambda_n$, what is the result of $\int_0^{\pi/2} y_m(x)y_n(x)dx$?

**E4.6** Expand $f(x) = x^2$ in terms of orthogonal set $\{\cos n\pi x\}_{n=0}^{\infty}$, $\{\sin n\pi x\}_{n=1}^{\infty}$, and $\{1,\, \cos n\pi x,\, \sin n\pi x\}_{n=1}^{\infty}$ in the interval $(-\pi,\, \pi)$. Thus, there are three expansions. Can we expand the function in the interval $(-1,\, 1)$ in terms of those functions? Why?

**E4.7** Check whether the following Sturm-Liouville eigenvalue problems are regular or singular; and whether the eigenvalues are positive or not. If the problem is singular, where is the singularity?

**(a).** $((1+x^2)y')' + \lambda y = 0, \quad y(0) = 0, \quad y(3) = 0.$

**(b).** $xy'' + y' + \lambda y = 0, \quad y(a) = 0, \quad y(b) = 0, \quad 0 \leq a < b.$

**(c).** $xy'' + 2y' + \lambda y = 0, \quad y(1) = 0, \quad y'(2) = 0.$ **Hint:** Multiply by $x$.

**(d).** $xy'' - y' + \lambda xy = 0, \quad y(0) = 0, \quad y'(5) = 0.$ **Hint:** Divide by $x^2$.

**(e).** $((1-x^2)y')' - 2xy' + (1+\lambda x)y = 0, \quad y(-1) = 0, \quad y(1) = 0.$

**Hint:** First you need to re-write the problems as the standard Sturm-Liouville eigenvalue problems if possible.

**E4.8** Find out the eigenvalues and eigenfunctions of the Sturm-Liouville eigenvalue problem. It is encouraged to use computers to plot first three eigenfunctions.

**(a).** $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(4\pi) = 0.$

**(b).** $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi/4) = 0.$

**(c).** $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(4\pi) = 0.$

**(d).** $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(4\pi) = 0.$

**(e).** $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(4\pi) = 0.$

**(f).** $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(4\pi) + y'(4\pi) = 0.$

**E4.9** Find out all eigenvalues and eigenfunctions of the Sturm-Liouville eigenvalue problem.

$$y'' + \lambda y = 0, \quad 0 < x < p,$$

$$y'(0) = 0, \quad y(p) = 0.$$

Plot first three eigenfunctions with $p = 1/2$, $p = 2$. Also answer the following questions:

**(a).** Can the eigenvalues of regular Sturm-Liouville eigenvalue problem be complex numbers?

**(b).** Are there finite or infinite number of distinct eigenvalues?

**(c).** If $y_m(x)$ and $y_n(x)$ are two eigenfunctions corresponding to two different eigenvalues $\lambda_m$ and $\lambda_n$, what is the result of $\int_0^p y_m(x)y_n(x)dx$ if $m \neq n$? How about $\int_0^{p/2} y_m(x)y_n(x)dx$?

**E4.10** Solve the 1D wave equation $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$ according to the following conditions:

**(a)**. The Cauchy problem $-\infty < x < \infty$ with $u(x,0) = xe^{-x}$, $\dfrac{\partial u}{\partial t}(x,0) = e^{-x}$.

**(b)**. The boundary value problem $u(0,t) = 0$, $u(2,t) = 0$, $0 < x < 2$ with $u(x,0) = \sin(4\pi x)$, $\dfrac{\partial u}{\partial t}(x,0) = \sin(4\pi x)$.

**(c)**. The boundary value problem $u(0,t) = 0$, $u(2,t) = 0$, $0 < x < 2$ with $u(x,0) = x$, $\dfrac{\partial u}{\partial t}(x,0) = x^2$.

**E4.11** Repeat the problem for the 1D heat equation $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ according to the following conditions:

**(a)**. The boundary value problem $u(0,t) = 0$, $u(2,t) = 0$, $0 < x < 2$ with $u(x,0) = \sin(4\pi x)$.

**(b)**. The boundary value problem $u(0,t) = 0$, $u(2,t) = 0$, $0 < x < 2$ with $u(x,0) = x$.

**E4.12** Solve the 1D wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$ with $\dfrac{\partial u}{\partial x}(0,t) = 0$, $u(L,t) = 0$ and the initial condition $u(x,0) = f(x)$, $\dfrac{\partial u}{\partial t}(x,0) = g(x)$. Find the solution when $c = 3$, $L = 2$, $g(x) = 1$, and $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{if } 1 \le x \le 2. \end{cases}$

**E4.13** Solve the 1D heat equation $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$ with $u(0,t) = 0$, $\dfrac{\partial u}{\partial x}(L,t) = 0$ and the initial condition $u(x,0) = f(x)$.

**(a)**. Let $u(x,t) = X(x)T(t)$, derive the equations for $X(x)$ and $T(t)$.

**(b)**. Solve the related Sturm-Liouville eigenvalue value problem for $X(x)$ first.

**(c)**. Solve for $T(t)$ using the eigenvalues above.

**(d)**. Find the series solution to the 1D heat equation.

**(e)**. Find the solution when $c = 3$, $L = 2$, and $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{if } 1 \le x \le 2. \end{cases}$

## Chapter 5

# Various Fourier series, properties and convergence

We have seen that $\left\{\sin \frac{n\pi x}{L}\right\}$ and $\left\{\cos \frac{n\pi x}{L}\right\}$ play very important roles in the series of solution of partial differential equations of boundary value problems by the method of separation of variables. While these orthogonal functions are obtained from Sturm-Liouville eigenvalue problems, they should have reminded us of Fourier series in which $\{\sin nx\}$ and $\{\cos nx\}$ are used. Fourier series have wide applications in many areas of sciences and engineering particularly in electro-magnetics, signal processing, filter design, and fast computation using fast Fourier transforms (FFT). In this chapter, we will introduce various Fourier series, discuss the properties and convergence of those series. We will see three kinds of Fourier expansions of a function $f(x)$:

**1** General Fourier expansions in $(-L, L)$

$$f(x) \sim \bar{a}_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right); \tag{5.1}$$

When $L = \pi$, we obtain the classical Fourier series. The reason to use $\bar{a}_0$ rather than $a_0$ can be seen later.

**2** Half-range sine expansions in $(0, L)$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}; \tag{5.2}$$

**3** Half-range cosine expansions $(0, L)$

$$f(x) \sim \bar{a}_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}. \tag{5.3}$$

## 5.1 Period, piecewise continuous/smooth functions

We know that $\sin x, \cos x, \sin 2x, \cos 2x, \cdots$, are all period functions. The above three types of Fourier expansions all involve sine and cosine functions that are periodic. What is a period function? A function repeats itself in a fixed pattern.

**Definition 5.1.** *If there is a positive number $T$ such at $f(x+T) = f(x)$ for any $x$, then $f(x)$ is called a period function with a period $T$.*

According to the definition, $f(x)$ should be defined in the entire space $(-\infty, \infty)$. Also, if $f(x) = f(x+T)$, then $f(x+2T) = f(x+T+T) = f(x+T) = f(x)$; and $2T$ is also a period of $f(x)$. To avoid the confusion, we only use the smallest such a $T > 0$, which is called the fundamental period, or simply the period, for short.

**Example 5.1.** *Find the period of* $\sin x, \cos x, \tan x,$ *and* $\cot x$.

The period of $\sin x, \cos x$ is $2\pi$, while the period of $\tan x, \cot x$ is $\pi$.

**Example 5.2.** *Are the following functions periodic? If so, find the period of the functions,*

$$\cos \pi x, \ \sin x + \tan x, \ \sin x + \cos \frac{x}{2}, \ \sin x + x, \cos mx.$$

**1** Yes, $\cos \pi x = \cos \pi(x+T) = \cos(\pi x + \pi T)$; so the period is $T = 2$.

**2** Yes, the sum of two periodic functions is still periodic; the period is the larger one, $T = 2\pi$.

**3** Yes, $\sin x + \cos \frac{x}{2} = \sin(x+T) + \cos \frac{x+T}{2}$. Since the period of the second function is $T/2 = 2\pi$, we conclude that the period is $T = 4\pi$.

**4** No, since $x$ is not a periodic function.

**5** Yes, from $\cos mx = \cos m(x+T) = \cos(mx + mT)$, we know that $mT = 2\pi$; thus the period is $T = \dfrac{2\pi}{m}$.

Note that if $f(x) = C$, then it is a periodic function of any period including the zero. From the above example, we also know that the set $\{1, \cos \frac{nx}{L}\}_{n=1}^{\infty}$ has a common period $T = 2L\pi$, the largest period of all $\cos \frac{nx}{L}$ for all $n$'s.

**Example 5.3.** *Let $f(x) = x - int(x) = x - [x]$, where $[x]$ is called a floor function,* *which means that* $[x]$ *is the greatest integer not larger than* $x$*, for example,* $[1.5] = 1$, $[0.5] = 0$, $[-1.5] = -2$*, or the integer toward left. Then* $f(x)$ *is a period function with period* $T = 1$*. The period function can be expressed as*

$$\begin{aligned} f(x) &= x, \qquad \text{if } 0 \le x < 1, \\ &= x - [x], \qquad \text{otherwise}, \end{aligned} \tag{5.4}$$

*or simply* $f(x) = f(x+1)$ *outside* $[0, 1]$*. Often it is enough to write down the function expression in one period and state that the function is periodic with the period specified. Figure 5.1 (a) is a plot of the integer (floor) function while Figure 5.1 (b) is a plot of the fraction part function that is a periodic with period* $T = 1$*.*

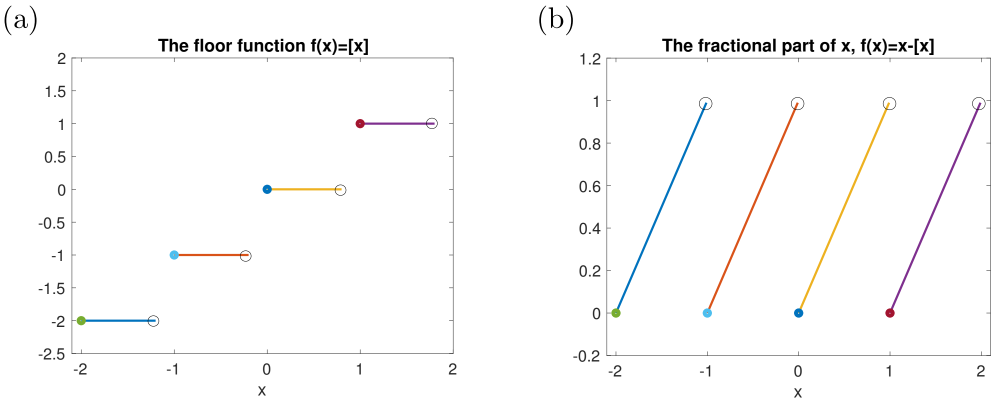



**Figure 5.1.** *Plots of some piecewise continuous functions. (a): the integer (floor) function that is not periodic. (b): the fractional part of* $x$ *that can be expressed as* $f(x) = x - int(x)$ *which is periodic with period* $T = 1$*.*

**Example 5.4.** *The sawtooth function is defined by*

$$f(x) = \begin{cases} \frac{1}{2}(-\pi - x) & \text{if } -\pi \le x < 0, \\ \frac{1}{2}(\pi - x) & \text{if } 0 \le x \le \pi, \end{cases} \tag{5.5}$$

*and* $f(x) = f(x + 2\pi)$*, see Figure 5.2 (b) for the function plot along its Fourier series and sum partial sums. Note that sometime it may be easier if we use the expression in the interval* $(0, 2\pi)$ *since it is one continuous piece as*

$$f(x) = \frac{1}{2}(\pi - x), \qquad 0 \le x < 2\pi,$$

*and* $f(x) = f(x+2\pi)$. *Figure 5.2 is a plot of the sawtooth function whose Fourier series and some partial sums are plotted in Figure 5.6.*

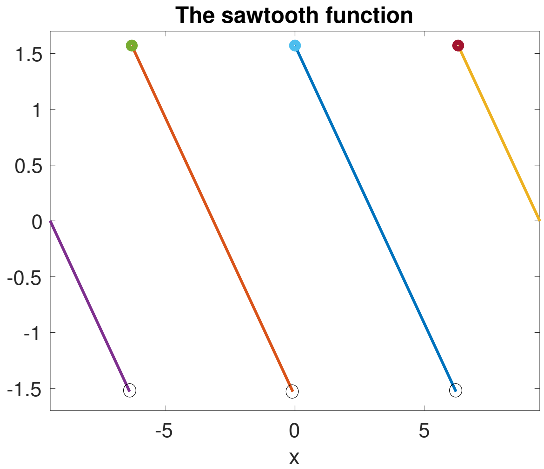



**Figure 5.2.** *Plots of the sawtooth function that is periodic with period* $T = 2\pi$*.*

## Piecewise continuous/smooth functions

If a function $f(x)$ is continuous in $[a,\ b]$, then for any $x_0 \in [a,\ b]$, we have $\lim_{x \to x_0} f(x) = f(x_0)$. We call that $f(x)$ is in the continuous function space (set with operations) denoted as $f(x) \in C[a,b]$. It is obvious that functions: $\sin x, \cos x, x^3+1$, and their linear combinations are continuous functions in any interval $[a,b]$. The functions $f(x) = 1/x$ is discontinuous at $x = 0$, but is continuous on any interval that does not contain the origin. Note that $1/x$ is continuous on $(0,\ 1]$ but not $[0,1]$. The function $\tan x$ is continuous on $[0,1]$ but not on $[0, \frac{\pi}{2}]$ since the left limit $\lim_{x \to \frac{\pi}{2}-} f(x) = \infty$. From these examples, we should see the difference between '[' (included) and '(' (not included) in describing an interval.

If there are *finite* number of points $x_1, x_2, \cdots, x_N$ in $[a,\ b]$ at which a function is not continuous, but has finite left and right limits, that is

$$\lim_{x \to x_i -} f(x) = f(x_i-) \quad \text{and} \quad \lim_{x \to x_i +} f(x) = f(x_i+)$$

exist but $f(x_i-)$ may not be the same as $f(x_i+)$, then such a function is called a piecewise continuous function in $(a,b)$, or precisely, a piecewise continuous and bounded function. Below is an example of a piecewise continuous and bounded function.

**Example 5.5.** *The Heaviside function* $H(x) = \begin{cases} 0 & \text{if } -\infty < x < 0, \\ 1 & \text{if } 0 \le x < \infty, \end{cases}$ *is a piecewise continuous function, which is also called a* step function.

Figure 5.1, the floor function and the fractional part function, and Figure 5.2, the sawtooth function, are some examples pf piecewise continuous functions. Pay attention to when we use little hollow o's, and little filled o's.

If $f(x)$ is a continuous function on an interval $(a, b)$, but $f'(x)$ is piecewise continuous on $(a, b)$, then $f(x)$ is called a piecewise smooth function on $(a, b)$. Below is an example.

**Example 5.6.** *The hat function* $h(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{otherwise,} \end{cases}$

is continuous but non-differentiable at $x = 0$ in the classical definition of derivatives. The derivative of the has function is

$$h'(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ 1 & \text{if } -1 < x < 0, \\ -1 & \text{if } 0 < x < 1, \end{cases}$$

which is discontinuous at $x = -1$, $x = 0$, and $x = 1$. It is obvious that the hat function is a piecewise smooth function, while $h'(x)$ is a piecewise continuous function, which is also a step function. Figure 5.3 shows a plot of the hat function at the left, and the derivative of the hat function at the right.

## Properties of period functions

The set of all period functions with the same period $T$ form a linear space. That is, let $f(x)$ and $g(x)$ be two period functions of period $T$, then $w(x) = \alpha f(x) + \beta g(x)$ is also a period function of period $T$. Note again that a period function is defined

(a) (b)

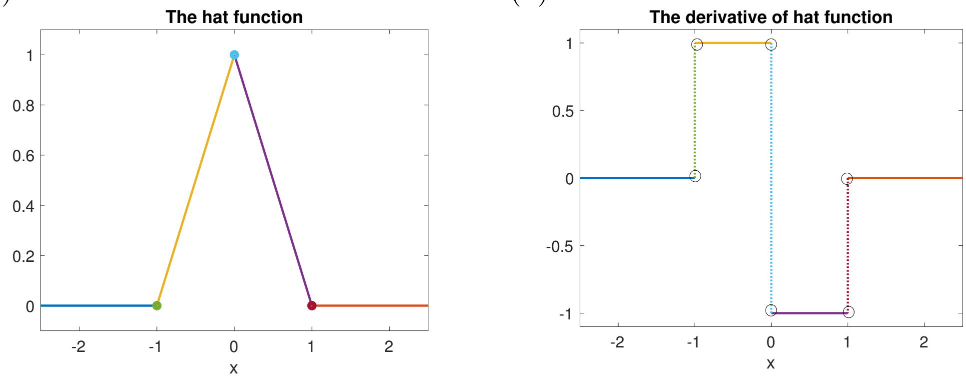



**Figure 5.3.** *Plot of the piecewise smooth hat function and its derivative. (a): the hat function; (b): the derivative of the hat function.*

in the entire space $-\infty < x < \infty$.

**Theorem 5.2.** *Let $f(x)$ be a period function of period $T$ and integrable, then*

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx \tag{5.6}$$

*for any real number $a$.*

**Proof:** To prove the theorem, we just need to show that $\int_a^{a+T} f(x)dx$ is a constant function of $a$. Therefore, we define $F(a) = \int_a^{a+T} f(x)dx$ and take the derivative with respect to $a$ to get,

$$\frac{dF(a)}{da} = f(a+T) - f(a) = 0.$$

Thus, $F(a)$ must be a constant, so $F(0) = F(a) = F(-T/2) = \cdots$, which leads to,

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx = \int_{a-\frac{T}{2}}^{a+\frac{T}{2}} f(x)dx.$$

Often we prefer to use the period that

- $f(x)$ is a continuous piece;
- integration starts from the origin ($a = 0$);
- integration from a symmetric interval $(-\frac{T}{2}, \frac{T}{2})$.

## 5.2 The classical Fourier series expansion and partial sums

Let $f(x)$ be a periodic function of $2\pi$ and $f(x) \in L^2(-\pi, \pi)$. The classical Fourier series expansion of $f(x)$ is defined as

$$f(x) \sim \bar{a}_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{5.7}$$

The coefficients $\bar{a}_0$, $\{a_n\}$ and $\{b_n\}$ are called the Fourier coefficients and can be computed from the following formulas,

$$\bar{a}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \tag{5.8}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, \qquad n = 1, 2, \cdots . \tag{5.9}$$

Note that $\bar{a}_0$ has different formula from other $a_n$'s by a factor of 2. We can use the same formula if we use

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{5.10}$$

We list a few applications of the Fourier series below:

- Express $f(x)$ in terms of simpler trigonometrical functions;
- Provide an approximation method for evaluating $f(x)$ using the partial sum defined as

  $$S_N(x) = \bar{a}_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx\right), \tag{5.11}$$

  as used in many computer packages for a given number $N$. We hope that $\lim\limits_{N\to\infty} S_N(x) = f(x)$;
- Basis for several fast algorithms such as Fast Fourier Transform (FFT);
- Used for spectral (frequency) analysis, signal processing, filters, etc.

Note that if $x$ is a time variable for some physical applications, we call that $f(x)$ is defined in the time domain, while $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are defined in the frequency domain.

**Classical Fourier series of $f(x) \in L^2(a,b)$:**

$$
\begin{aligned}
f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right), \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\end{aligned} \tag{5.12}
$$

**Example 5.7.** *Find the classical Fourier series of $f(x) = x$.*

**Solution:** We may wonder $f(x) = x$ is not a periodic function and why it can have a Fourier expansion. In fact, we only use part of $f(x) = x$ in the interval $(-\pi, \pi)$ and disregard the rest (*truncation* ). We then use the piece of $f(x) = x$ in the interval $(-\pi, \pi)$ to generate a periodic function (*extension*),

$$
\tilde{f}(x) = \begin{cases} x & \text{if } |x| \le \pi, \\ \tilde{f}(x + 2\pi) & \text{otherwise,} \end{cases}
$$

to get a periodic function that is identical to $f(x)$ in the interval $(-\pi, \pi)$, see Figure 5.4 for an illustration. The function $\tilde{f}(x)$ is piecewise continuous and bounded with discontinuities at $x = \pm 2n\pi, n = 1, 2, \cdots$. We use the formula to calculate the Fourier coefficients,

$$
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0, \quad n = 1, 2, \cdots,
$$

since $f(x)$ and $f(x) \cos nx$ are odd functions. Furthermore, we have

$$
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{\pi} \int_{0}^{\pi} x \sin nx dx = -\frac{2}{n\pi} x \cos nx \bigg|_0^{\pi} = (-1)^{n+1} \frac{2}{n}.
$$

Thus, we get

$$
x \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx = 2 \sin x - \sin 2x + \frac{2}{3} \sin 3x - \frac{1}{2} \sin 4x + \cdots.
$$

From the formula (5.12), we can have the Fourier series for any function $f(x)$ on $(-\pi, \pi)$ literally as long as those integrations in the formula are finite. But the series may or may not converge, or converge to a value that is different from $f(x)$.

To discuss the convergence of a Fourier series, we use the partial sums defined as before,

$$S_N(x) = f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)\,. \tag{5.13}$$

If the limit $\lim_{N\to\infty} S_N(x^*) = S(x^*)$ exists, then $S(x^*)$ is defined as the value of the series at $x^*$.

In Figure 5.4 (a), we plot the function $f(x) = x$ and a few partial sums of its Fourier series, $S_1(x)$, $S_5(x)$, $S_{55}(x)$ using the Maple. Figure 5.4 (b) is the function plot and the series plot in the interval $(-3\pi, 3\pi)$. From the figure, we can see that the partial sum $S_N(x)$ has the following properties:

**1** converges to $f(x)$ in the interior of $(-\pi, \pi)$ as $N \to \infty$;

**2** its value at $x = \pm\pi$ is not the left or right limit of $\tilde{f}(x)$, rather than its average, for example, at $x = \pi$,

$$\lim_{N\to\infty} S_N(\pi) = \frac{\tilde{f}(\pi-) + \tilde{f}(\pi+)}{2} = 0; \tag{5.14}$$

**3** $S_N(x)$ oscillates at the discontinuities $\pm 2n\pi$, $n = 1, 2, \cdots$. It is called the Gibb's phenomenon.

Note that the series itself is not oscillatory and it is identical to $f(x) = x$ in the interval $(-\pi, \pi)$, and it is zero at $x = \pm\pi$ which is the average of the left and right limits of the new period function $\tilde{f}(x)$, the *same* as the value of partial sums at $x = \pm\pi$, see Figure 5.4 (b).

**Example 5.8.** *Find the classical Fourier series of* $f(x) = 10\sin x + 5\sin 6x + \frac{1}{2}\cos 30x$.

**Solution:** The Fourier series of $f(x)$ is itself with $a_{30} = 0.5$, $b_1 = 10$, $b_6 = 5$. In Figure 5.5, we plot of the function and can see clearly the three frequencies and their strengths that agree with the function.

**Example 5.9.** *Find the Fourier series of the sawtooth function* $f(x) = \frac{1}{2}(\pi - x)$, $0 \le x < 2\pi$, $f(x + 2\pi) = f(x)$.

Note that $f(x)$ is an odd function in the interval of $(-\pi, \pi)$, thus $a_n = 0$, for $n = 0, 1, \cdots$. For the coefficients of $b_n$, it is easier to use one continuous piece in

(a) (b)

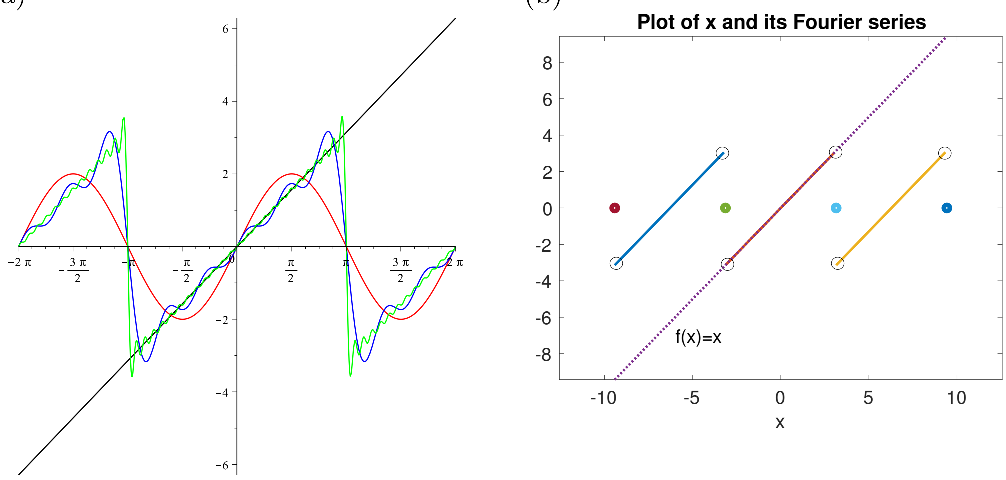



**Figure 5.4.** *(a): Plots of function $f(x) = x$ and some partial sums $S_1(x)$, $S_5(x)$, $S_{55}(x)$ of the Fourier series of the function. (b): Plots of the Fourier series of $\tilde{f}(x)$ whose values at $x = \pm 3\pi, \pm\pi$ are zero, and $f(x) = x$, the dotted line. Note that the two functions are identical in $(-\pi, \pi)$.*

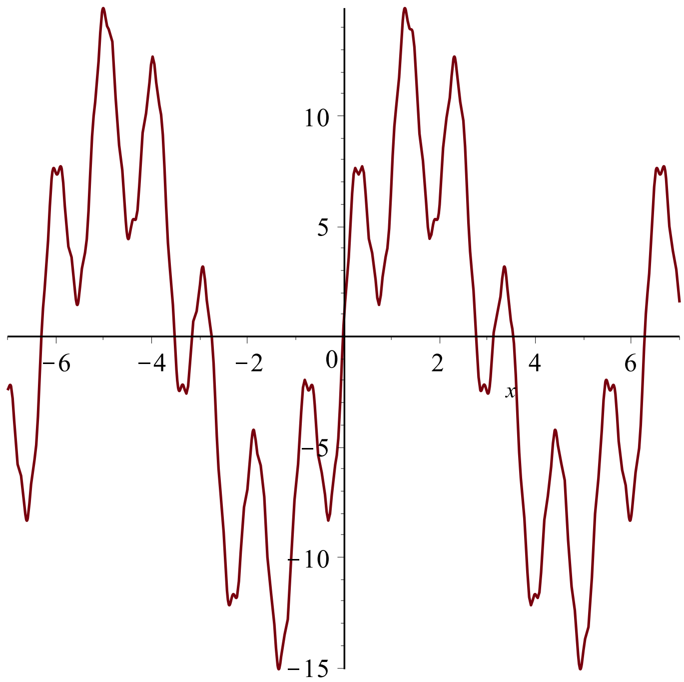



**Figure 5.5.** *Plot of $f(x) = 10\sin x + 5\sin 6x + \frac{1}{2}\cos 30x$. We can see clearly three different frequencies.*

$(0, 2\pi)$ to calculate the coefficients $b_n$,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\sin nx\,dx \\ &= \frac{1}{2\pi}\left(\int_0^{2\pi}\pi\sin nx\,dx + \frac{x\cos nx}{n}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{x\cos nx}{n}\,dx\right) \\ &= \frac{1}{2\pi}\frac{2\pi}{n} = \frac{1}{n}. \end{aligned}$$

Thus, the Fourier series is, see also Figure 5.2 (a),

$$\frac{1}{2}(x-\pi) \sim \sum_{n=1}^{\infty}\frac{\sin nx}{n} = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + c\cdots + \frac{\sin nx}{n} + \cdots.$$

In Figure 5.2, we plot the function $f(x) = (\pi - x)$ of period $2\pi$ and a few partial sums. We observe that the partial sum converges to $f(x) = x - \pi$ except at those discontinuities at $x = 0$ and $x = \pm\pi$. Once again, we see the Gibb's oscillations of the partial sums around the discontinuities. It is also important to note that the series converges to $f(x)$ in the interval except at the discontinuities where the value of the series is the average of the left and right limits which is zero in this case. *There is no oscillations in the Fourier series though!*

(a) (b)

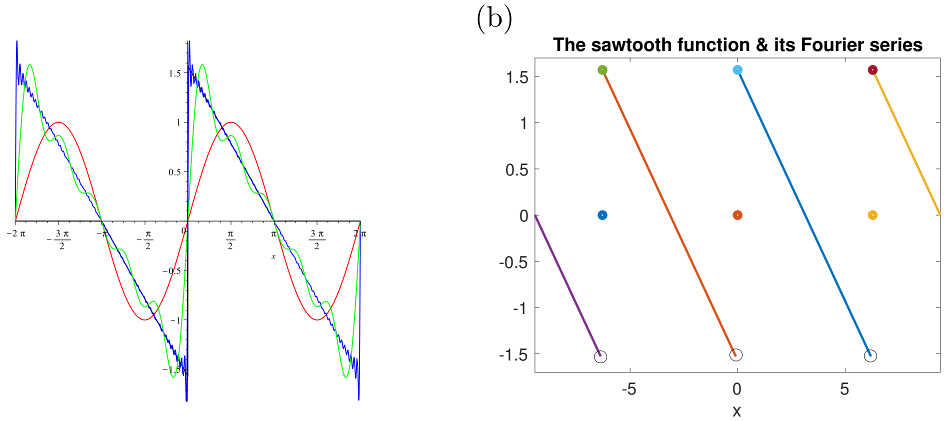



**Figure 5.6.** *(a): Plots of some partial sum of the Fourier series of the sawtooth function. (b): Plot the sawtooth function and its Fourier series. They are identical except at the discontinuities where the series is the average of the left and right limits.*

**Example 5.10.** *Find the Fourier series of the triangular wave.*

$$f(x) = \begin{cases} x + \pi & \text{if } -\pi \le x < 0, \\ \pi - x & 0 \le x \le \pi, \end{cases} \tag{5.15}$$

*and $f(x) = f(x + 2\pi)$. Note that we can rewrite the function as one piece as $f(x) = \pi - |x|$ in the interval $(-\pi, \pi)$, which is an even function in $(-\pi, \pi)$. The function is continuous in the interval and it is piecewise smooth.*

**Solutions:** Note that $f(x)$ is an even function, we have

$$\begin{aligned}
\bar{a}_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)dx = -\frac{1}{\pi}\frac{(\pi - x)^2}{2}\bigg|_0^{\pi} = \frac{\pi}{2} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx dx \\
&= \frac{2}{\pi}\left(\frac{(\pi - x)\sin nx}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx dx\right) \\
&= \frac{2}{\pi}\left(\frac{-\cos nx}{n^2}\bigg|_0^{\pi}\right) = \frac{2}{\pi}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right) \\
&= \frac{2}{\pi}\begin{cases} \dfrac{2}{n^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

We can use one simple notation $a_{2k+1} = \frac{4}{\pi(2k+1)^2}$ to cover both situations. Thus, we have,

$$f(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{4}{\pi(2n + 1)^2}\cos \pi(2n + 1)x.$$

Since $f(x)$ is continuous everywhere, we have the equality in the entire interval! In Figure 5.7, we plot the function of the triangular wave $f(x) = \pi - |x|$ of period $2\pi$, and a few partial sums. We observe that the partial sum converges to $f(x) = \pi - |x|$ in the entire domain. We do not see the Gibb's oscillations but round-ups at the kinks, $x = 0$ and $x = \pm\pi$.

We can get some identities from the Fourier series. In this example, we have

$$f(0) = \pi = \frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{4}{\pi(2n + 1)^2}, \tag{5.16}$$

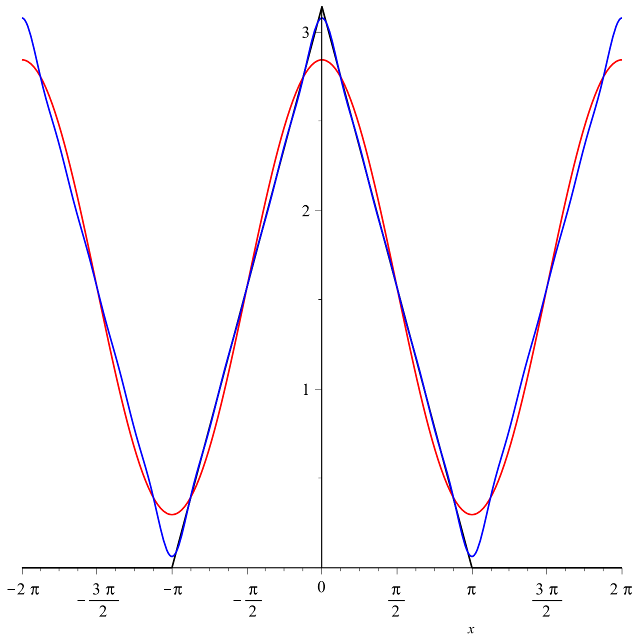



**Figure 5.7.** *Plot of the triangular wave and several partial sums. There is no Gibb's phenomenon but round-up at the kinks. The series is identical to the function of the triangular wave.*

which provide an alternative way in computing $\pi$ if we multiply $\pi$ from both sides and simplify to get

$$\frac{\pi^2}{2} = \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2} \qquad \Longrightarrow \qquad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{7^2} + \cdots.$$

## 5.3 Fourier series of functions with arbitrary periods

Given a period function $f(x) \in L^2(-L, L)$ with $f(x+T) = f(x)$ and $T = 2L$, we can also have a Fourier series expansion of $f(x)$ in $(-L,\ L)$. To derive the Fourier series for a periodic function of $2L$, we use a linear transform to convert the interval $(-L,\ L)$ to $(-\pi,\ \pi)$, apply the Fourier expansion, and then transform back using the original variable.

Let $t = \alpha x$ and $\alpha$ is chosen such that when $x = -L$, $t = -\pi$, and when $x = L$, $t = \pi$. It is easy to get $\alpha = \frac{\pi}{L}$. Define also $f(x) = f(\frac{t}{\alpha}) = F(t)$. We can verify that $F(t)$ is a period function of $2\pi$ since

$$F(t+2\pi) = f\left(\frac{t+2\pi}{\alpha}\right) = f\left(\frac{t}{\alpha} + \frac{2\pi}{\alpha}\right) = f\left(\frac{t}{\alpha} + 2L\right) = f\left(\frac{t}{\alpha}\right) = F(t).$$

Thus, $F(t)$ has the Fourier series,

$$F(t) \sim \bar{a}_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

$$\bar{a}_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) dt, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \cos nt\, dt,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \sin nt\, dt, \qquad n = 1, 2, \cdots.$$

By changing the variable again using $t = \frac{\pi}{L}$ in all the expressions above, we get

**Fourier Series of $f(x)$ with an Arbitrary Period $2L$:**

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{5.17}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx.$$

In the expression above we have $a_n$, $n = 0, 1, \cdots, \cdots$, and $b_n$, $n = 1, 2, \cdots$. Note that the above formula includes the classical Fourier series if we take $L = \pi$. Thus, it is enough just to remember this formula. Again the partial sum of the Fourier expansion in $(-L, L)$ is defined as

$$S_N(x) = \bar{a}_0 + \sum_{n=1}^{N} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{5.18}$$

for a positive integer $N > 0$.

**Example 5.11.** *Recall that the fractional part of $x$ is a periodic function of period $T = 1$ and $p = 1$. The function can be written as $f(x) = x - int(x)$. We can find its Fourier series in $(-1, 1)$.*

Using the formula, we have

$$\bar{a}_0 = \frac{1}{2} \int_{-1}^{1} f(x) dx = \frac{1}{2} \left( \int_{-1}^{0} (x+1) dx + \int_{0}^{1} x dx \right) = \frac{1}{2},$$

$$a_n = \int_{-1}^{1} f(x) \cos \frac{n\pi x}{p} dx = \int_{-1}^{0} (x+1) \cos n\pi x\, dx + \int_{0}^{1} x \cos n\pi x\, dx = 0,$$

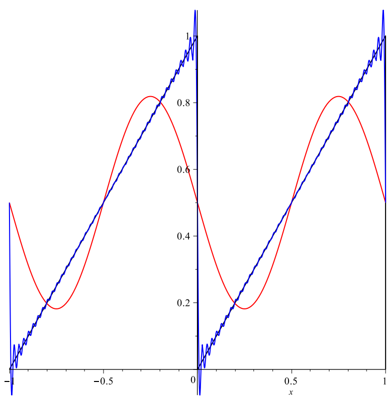
$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(x) \sin \frac{n\pi x}{p} dx = \int_{-1}^{0} (x+1) \sin \frac{n\pi x}{p} dx + \int_{0}^{1} x \sin \frac{n\pi x}{p} dx \\
&= \int_{-1}^{0} \sin \frac{n\pi x}{p} dx + 2\int_{0}^{1} x \sin \frac{n\pi x}{p} dx = -\frac{1}{n\pi}\left(1-(-1)^n\right) + \frac{1}{n\pi}(-1)^n \\
&= \begin{cases} 0 & \text{if } n = 2k+1, \\ -\dfrac{2}{(n\pi)} & \text{if } n = 2k, \end{cases}
\end{aligned}
$$

Thus, we obtain

$$
f(x) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(2k\pi x).
$$

In Figure 5.8, we plot the function $f(x) = x - int(x)$ and several partial sums of the Fourier series. The Fourier series converges to $f(x)$ in the interior. We observe the Gibb's oscillations at the discontinuity at $x = 0$, and the two end points $x = -1$ and $x = 1$.

(a)



(b)

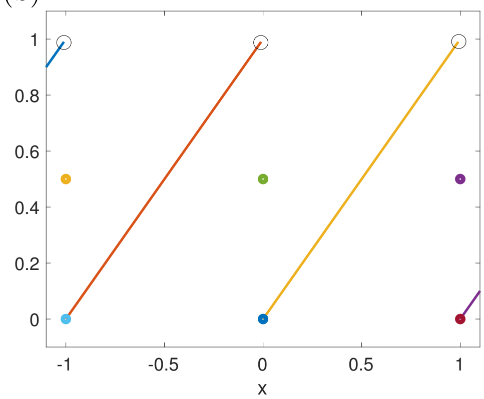



**Figure 5.8.** *(a): Plots of some partial sum of the Fourier series of the fraction part function, $f(x) = x - int(x)$. (b): Plot the function and its Fourier series. They are identical except at the discontinuity $x = 0$ and two end points $x = \pm 1$ where the series is the average of the left and right limits of the periodic function.*

**Remark 5.1.** *For the fractional part function example, the period is $T = 1$, we can also have the Fourier series in $(-\frac{1}{2}, \frac{1}{2})$ which will be different from that in $(-1, 1)$.*

**Example 5.12.** *Expand $f(x) = |x|$ in Fourier series in $(-p, p)$ for a parameter $p > 0$.*

**Solution:** Note that $f(x)$ is an even function, we have

$$\bar{a}_0 = \frac{1}{2p}\int_{-p}^{p} |x| dx = \frac{2}{2p}\int_0^p x dx = \frac{1}{p}\frac{x^2}{2}\bigg|_0^p = \frac{p}{2},$$

$$\begin{aligned} a_n &= \frac{1}{p}\int_{-p}^{p} |x| \cos\frac{n\pi x}{p} dx = \frac{2}{2p}\int_0^p x\cos\frac{n\pi x}{p} dx \\ &= -\frac{2p}{(n\pi)^2}(1-\cos n\pi) = \begin{cases} 0 & \text{if } n = 2k, \\ -\dfrac{4p}{(n\pi)^2} & \text{if } n = 2k+1, \end{cases} \\ b_n &= \frac{1}{p}\int_{-p}^{p} |x| \sin\frac{n\pi x}{p} dx = 0, \end{aligned}$$

since $f(x)$ and $f(x)\cos\frac{n\pi x}{p}$ are even functions, and $f(x)\sin\frac{n\pi x}{p}$ is an odd function. Thus, we obtain

$$\begin{aligned} |x| &= \frac{p}{2} - \sum_{n=0}^{\infty} \frac{4p}{((2n+1)\pi)^2}\cos\frac{(2n+1)\pi x}{p} \\ &= \frac{p}{2} - \frac{4p}{\pi^2}\left(\cos\frac{\pi x}{p} + \frac{1}{3^2}\cos\frac{3\pi x}{p} + \frac{1}{5^2}\cos\frac{5\pi x}{p} + \frac{1}{7^2}\cos\frac{7\pi x}{p} + \cdots\right). \end{aligned}$$

In Figure 5.9, we take $p = 1$ and plot the function $f(x)$ and several partial sums of the Fourier series in the interval $(-2, 2)$. The Fourier series converges to $|x|$ only in the interval $[-1, 1]$ including the two end points. No Gibb's phenomenon is present for the partial sums since the function is piecewise smooth. But we do see that the kink of $|x|$ at $x = 0$ is smoothed by the partial sums, which are called *round-ups*.

When $p = \pi$, we get the classical Fourier series in $[-\pi, \pi]$,

$$\begin{aligned} |x| &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi}\cos(2n+1)x \\ &= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \frac{1}{7^2}\cos 7x + \cdots\right). \end{aligned}$$

**Remark 5.2.** *In the expansion above, we expand the $2p$-function $f(x) = |x|$ and $f(x + 2p) = f(x)$ in terms of the Fourier series. The expansion is the same as*

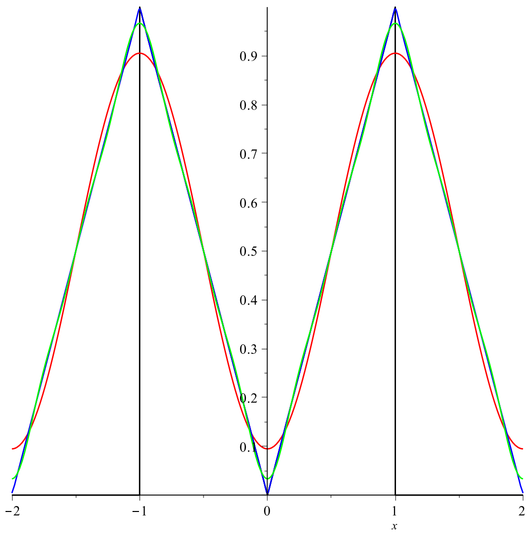



**Figure 5.9.** *Plot of the Fourier series and several partial sums of* $|x|$ *in the interval* $(-2, 2)$ *with* $p = 1$*. The Fourier series converges to* $|x|$ *only in the interval* $[-1, 1]$.

*that for function* $g(x) = |x|$ *in the interval* $(-p, p)$ *but totally different outside the interval. There is no Gibb's phenomenon and the series is convergent to* $|x|$ *in* $(-p, p)$*. The process can be summarized as* extension and expansion.

**Example 5.13.** *Expand* $f(x) = \sin x$ *in Fourier series in* $(-p, p)$ *for a parameter* $p > 0$.

**Solution:** If $p = \pi$, then the Fourier expansion of $\sin x$ is itself. Otherwise, we can expand $\sin x$ in terms of $\sin \frac{n\pi x}{p}$. Note that $a_n = 0$, $n = 0, 1, \cdots$, since $f(x)$ is an odd function. We just need to find $b_n$,

$$b_n = \frac{1}{p}\int_{-p}^{p} \sin x \sin \frac{n\pi x}{p} dx = \frac{2}{2p}\int_{0}^{p} \sin x \sin \frac{n\pi x}{p} dx$$
$$= \frac{2p\,(n\pi \sin p\, \cos n\pi - p \cos p\, \sin(n\pi))}{p^2 - \pi^2 n^2}.$$

The integration is obtained by using the formula

$$\sin\alpha \sin\beta = -\frac{1}{2}\left(\cos(\alpha+\beta) - \cos(\alpha-\beta)\right)$$

or using the Maple command

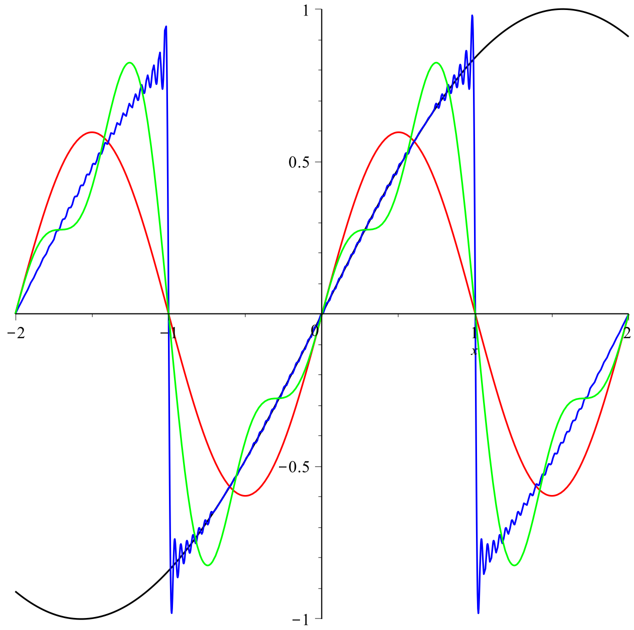



**Figure 5.10.** *Plots of several partial sums of the Fourier series of* $\sin x$ *in the interval* $(-2, 2)$ *with* $p = 1$. *The Fourier series converges to* $\sin x$ *only in the interval* $(-1, 1)$.

```
\int_0^p \sin x \sin \frac{n \pi x}{p} dx;
```

For the special case $p = 1$, we can get

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n\pi \sin 1 \sin(n\pi x)}{n^2\pi^2 - 1},$$

which is valid only in the interval $(-1, 1)$, see Figure 5.10 for an illustration.

It is important to know the relations among the function itself, its Fourier series, and the partial sums. If the function $f(x)$ is continuous at a point $x^*$, then the series has the *same value* as that of $f(x)$. The partial sums are *approximations* to $f(x)$ and are different from $f(x)$ in general, that is, $f(x^*) \neq S_N(x^*)$. Nevertheless, the limit of the partial sum is $f(x^*)$, that is, $\lim_{N\to\infty} S_N(x^*) = f(x^*)$ if $f(x)$ is continuous at $x^*$. If $f(x)$ is discontinuous at a point $x^*$, then the value of the series is the average of the left and right limit, that is

$$\lim_{N\to\infty} S_N(x^*) = S(x^*) = \frac{\lim\limits_{x\to x^*, x<x^*} f(x) + \lim\limits_{x\to x^*, x>x^*} f(x)}{2} = \frac{f(x^*-) + f(x^*+)}{2}.$$

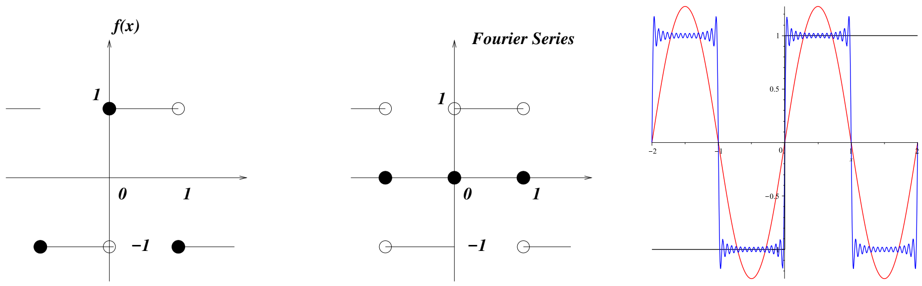



**Figure 5.11.** *Plots of $f(x)$, its Fourier series, and partial sums. Left: $f(x)$; Middle: Fourier series; Right: Two partial sums with $N = 2$, $N = 30$. Gibb's oscillations can been seen at the discontinuities if $N$ is large enough.*

We use a step function below to illustrate the relations,

$$f(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0, \\ 1 & 0 \leq x < 1. \end{cases} \tag{5.19}$$

In Figure 5.11, we plot the step function in the left; the Fourier series of the function in the middle; and two partial sums of the Fourier series in the right. We can see that $f(x)$ is piecewise continuous. The Fourier series is identical to $f(x)$ except at the discontinuities $x = -1$, $x = 0$, and $x = 1$. At these points, the Fourier series is the average of the left and right limit of the function, for example,

$$S(0) = \frac{f(0-) + f(0+)}{2} = \frac{-1+1}{2} = 0, \tag{5.20}$$

which is the same at $x = -1$ and $x = 1$. The partial sums are not the same as $f(x)$ but they will get closer to $f(x)$ as $N$ increases except at those discontinuities at which the values are also the average of the left and right limit of the function. Note also that we will see the Gibb's oscillations around the discontinuities if $N$ is large enough. Intuitively, the Fourier series tries to approximate both the left and right limit, which is impossible and causes the oscillations.

## 5.4 Half-range expansions

We have already seen that we can choose different expansions and seen some connections between Fourier series and orthogonal functions from the theory of Sturm-Liouville eigenvalue problems. With half-range expansion, we can also reduce some workload compared to a full range expansion, and impose some special properties of the expansions. The techniques once again is based some particular truncations and extensions.

Let $f(x)$ be a piecewise continuous function in $(0, L)$[4]. If we extend $f(x)$, $0 \leq x \leq L$ in the following way,

$$f_e(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L, \\ f(-x) & \text{if } -L < x < 0, \end{cases} \tag{5.21}$$

which is called an *even extension* of $f(x)$, then we can have the Fourier series expansion of $f_e(x)$ in the interval $(-L, L)$. Since $f_e(x)$ is an even function, we have $b_n = 0$ and the expansion has cosine functions only

$$\bar{a}_0 = \frac{1}{2L}\int_{-L}^{L} f_e(x)dx = \frac{1}{L}\int_0^L f(x)dx, \tag{5.22}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{5.23}$$

Also in the interval, we have $f_e(x) = f(x)$, thus we obtain:

**Half Range Cosine Series Expansion of $f(x)$ in $(0, L)$:**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \tag{5.24}$$

**for $n = 0, 1, 2, \cdots$. The expansion is valid only in $(0, L)$.**

Similarly, if we extend $f(x)$ , $0 \leq x \leq L$ according to

$$f_o(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L, \\ -f(-x) & \text{if } -L < x < 0, \end{cases} \tag{5.25}$$

which is called an *odd extension* of $f(x)$, then we can have the Fourier series expansion of $f_e(x)$ in the interval $(-L, L)$. Since $f_o(x)$ is an odd function, we have $a_n = 0$ and the expansion has sine functions only

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \tag{5.26}$$

[4]In fact, the discussions are valid for any interval $(a, b)$ $(b > a)$. we can use a shift $s = x - a$ to change the domain from $(a, b)$ in $x$ to $(0, b - a)$ in terms of $s$.

$n = 1, 2, \cdots,$. Also in the interval, we have $f_o(x) = f(x)$, thus we have

**Half Range Sine Series Expansion of $f(x)$ in $(0, L)$:**

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad (5.27)$$

**for $n = 1, 2, \cdots$. The expansion is valid only in $(0, L)$.**

**Example 5.14.** *Expand $f(x) = x$ in both half range cosine and sine series in $(0, 1)$. What is the relation of the expansion with the Fourier series in $(-1, 1)$.*

**Solution:** The function $f(x)$ is an odd function. Thus, the half range sine series is the same as the Fourier series in $(-1, 1)$ for which the coefficients $a_n = 0$ and $b_n$ can be calculated as (verified by Maple),

$$b_n = 2 \int_0^1 x \sin(n\pi x) dx = -2 \frac{\cos n\pi}{n\pi} = (-1)^n \frac{2}{n\pi}.$$

Thus, the sine expansion (and the Fourier expansion) of $f(x) = x$ in the interval $(0, 1)$ is

$$x = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n\pi} \sin(n\pi x), \qquad x \in (0, 1).$$

The series is convergent in the interior of $[0, 1)$ but is zero at $x = 1$ ($f(1) \neq S(1)$), see Figure 5.12 (b) for plots of the function, and several partial sums of the expansion. Note that the partial sums $S_N(x)$ have Gibb's oscillations near $x = 1$ if $N$ is large enough.

For the cosine half range expansion, the expansion is only valid in $[0, 1]$, we have

$$a_0 = \int_0^1 x dx = \frac{1}{2}, \qquad a_n = 2 \int_0^1 x \cos(n\pi x) dx = 2 \frac{\cos n\pi - 1}{(n\pi)^2} = \frac{-4}{(2k-1)^2 \pi^2}.$$

Thus, the cosine expansion of $f(x) = x$ in the interval $(0, 1)$ can also be represented as

$$x = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi x}{(2n-1)^2}.$$

The series is convergent in the entire interval of $[0, 1]$, see Figure 5.12 (a) for plots of the function and several partial sums of the expansion. In this case, we have faster convergence of the partial sum of the cosine expansion compared with the sine expansion. Note that the partial sums $S_N(x)$ do not have Gibb's oscillations but round-ups near $x = 1$ if $N$ is large enough.

(a) (b)

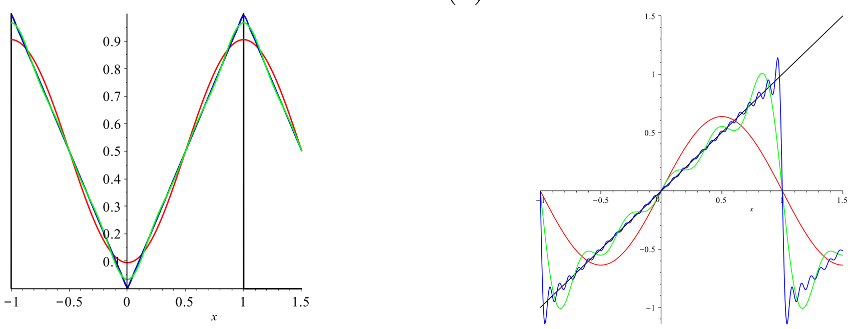



**Figure 5.12.** *Half range cosine/sine Fourier expansions of $f(x) = x$ in $(0, 1)$ and plots of several partial sums. (a) Half cosine, the series is convergent in $[0, 1]$; (b) Half sine, the series is convergent in $[0, 1)$ but not to $x$ at $x = 1$.*

**Example 5.15.** *Expand $f(x) = \cos x$ in both half range cosine and sine series in $(0, \pi)$. What is the relation of the expansion with the Fourier series in $(-\pi, \pi)$. How about in $(0, 1)$?*

**Solution:** The function $f(x) = \cos x$ is an even function. Thus, the half range cosine series is the same as the Fourier series in $(-\pi, \pi)$ or any $2\pi$ intervals, which is itself but it is different in $(-1, 1)$.

For the half-range sine expansion, we have (verified by Maple)

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin(nx) dx = \frac{2}{\pi} \frac{n(\cos n\pi + 1)}{n^2 - 1} = \frac{1}{\pi} \frac{8k}{(2k)^2 - 1},$$

where $n = 2k$ since for odd $n$'s, $b_n = 0$. Thus, the sine expansion of $f(x) = \cos x$ in the interval $(0, \pi)$ is

$$\cos x = \sum_{n=1}^{\infty} \frac{1}{\pi} \frac{8n}{4n^2 - 1} \sin(2nx), \qquad x \in (0, \pi).$$

The series is convergent in the interior of $(0, \pi)$ but not to $\cos x$ at $x = 0$ and $x = \pi$, see Figure 5.13 (a) for plots of the function, and several partial sums of the expansion. Note that the partial sums $S_N(x)$ have Gibb's oscillations near $x = 0$ and $x = \pi$ if $N$ is large enough.

We also plot the function and several partial sums of the cosine expansion of $\cos x$ in $(0, 1)$. In this case, the series is convergent in the entire interval $[0, 1]$ including two end points. The partial sums $S_N(x)$ have round-ups at $x = 1$ if $N$ is large enough but no Gibb's oscillations .

(a) (b)

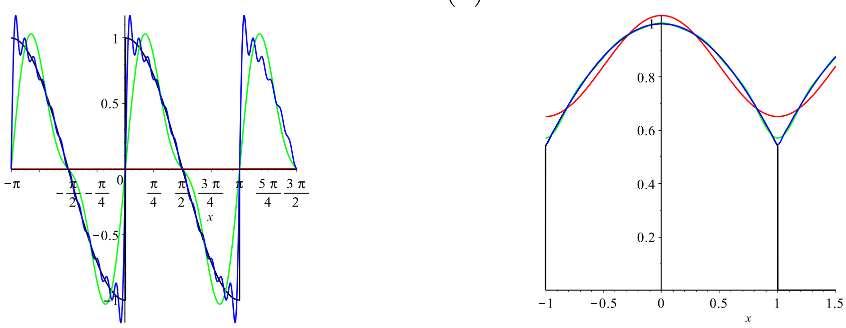



**Figure 5.13.** *Illustration of half range sine/cosine Fourier expansions of* $f(x) = \cos x$. *(a): plots of the function, several partial sums of the half range sine expansion on* $(0, \pi)$. *The series is convergent to* $f(x) = \cos x$ *in* $(0, \pi)$ *but not at two ends; (b): half range cosine on* $(0, 1)$. *The series is convergent to* $f(x) = \cos x$ *on* $[0, 1]$ *including the two ends.*

## 5.5 Some theoretical results of various Fourier series

First of all, from the orthogonality of $\left\{\cos \frac{n\pi x}{L}\right\}_{n=0}^{\infty}$ and $\left\{\sin \frac{n\pi x}{L}\right\}_{n=1}^{\infty}$, we can easily prove the Parseval's identity.

**Parseval's identity:** If $f(x) \in L^2(-L, L)$ and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \quad -L < x < L,$$

then the following Parseval's identity holds

**Parseval's Identity**

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right). \qquad (5.28)$$

Note that the identity is not true for the integration in any interval but only for $(-L, L)$ for which the trigonometric functions on the right hand side is orthogonal!

**Example 5.16.** *If* $f(x) = \sum_{n=0}^{\infty} \frac{\cos nx}{2^n}$, *find the value of* $\int_{-\pi}^{\pi} f^2 dx$.

**Solution:** In this example, $L = \pi$, $a_0 = 2$, $a_n = \frac{1}{2^n}$, $b_n = 0$, thus we have

$$\frac{1}{2\pi} \int_{-L}^{L} |f(x)|^2 dx = 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{2n}} = 1 + \frac{1}{2} \left( \frac{1/4}{1 - 1/4} \right) = \frac{7}{6}$$

$$\Longrightarrow \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{7\pi}{3}.$$

From Parseval's identity, we can get some useful identities of series like the one above $\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{2n}}$.

Now we discuss the calculus of Fourier series, which often deals with the limits, the differentiation, and integration of Fourier series. The tool is to use the partial sum of a series. We want to know whether the following is true.

$$\left( \lim_{x \to x_0}; \ \frac{d}{dx}; \ \int_{\alpha}^{\beta} dx \right) f(x) \stackrel{?}{=} \sum_{n=0}^{\infty} \left( \lim_{x \to x_0}; \ \frac{d}{dx}; \ \int_{\alpha}^{\beta} dx \right) \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

The partial sum forms a sequence $\{S_0(x), S_1(x), S_2(x), \cdots, S_N(x), \cdots\}$ or $\{S_N(x)\}$. Note that $S_N(x)$ has two parameters, $x$ and $N$. We will discuss two kinds of convergence, pointwise and uniform convergence in an interval. We will discuss more general sequence $f_n(x)$.

A pointwise convergence of $f_n(x)$ is defined for a fixed point $x$ in an interval $(a, b)$ such that $\lim_{n \to \infty} f_n(x) = f(x)$. For the partial sum $S_N(x)$, the pointwise convergence is the same as the convergence of the series.

**Example 5.17.** *Are the following sequences convergent? (a),* $f_n(x) = \frac{\sin nx}{n}$; *(b),* $g_n(x) = nxe^{-nx+1}$.

**Solution:** (a), $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{\sin nx}{n} = 0$ for any $x$. (b), we can use the I'Hospital's rule to get the limit, that is, $\lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} \frac{nxe}{e^{nx}} = \lim_{n \to \infty} \frac{xe}{xe^{nx}} = 0$ for any $x \neq 0$, in which we differentiate $n$ in the I'Hospital's rule.

In above examples, both $f_n(x)$ and $g_n(x)$ are convergent to zero in any interval. The function $f_n(x)$ gets smaller and smaller as $n$ gets large, while there are always points $x$ near zero such that $g_n(x) \sim 1$ no matter how large $n$ can be. Such an $f_n(x)$ is also called uniformly convergent, while $g_n(x)$ is not uniformly convergent in the interval $(0, \pi)$. Note that $g_n(x)$ is uniformly convergent in any interval $(a, b)$ if $a > 0$.

**Definition 5.3.** *Let $f_n(x)$ be a sequence defined in an interval $[a, b]$ and $f_n(x)$ has the pointwise convergence $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x$ in $[a, b]$. Given any number $\epsilon > 0$ (no matter how small it may be), if there is an integer $N$ such that*

$$|f_n(x) - f(x)| < \epsilon \quad \text{for any } n > N \text{ and } x \text{ in } [a, b], \tag{5.29}$$

*then $f_n(x)$ is called uniformly convergent to $f(x)$ in $[a, b]$.*

In the previous example, given an $\epsilon > 0$, for $f_n(x) = \frac{\sin nx}{n}$, we have

$$|f_n(x)| = \left|\frac{\sin nx}{n}\right| \le \left|\frac{1}{n}\right| < \epsilon,$$

as long as $n \ge \text{int}(1/\epsilon) + 1$. Thus, we can take $N = \text{int}(1/\epsilon) + 1$ after which we have $|f_n(x) - 0| < \epsilon$. However, for $g_n(x) = nxe^{-nx+1}$, no matter how large $n$ is, we can find an $x = 1/n$ for which $g_n(x) = 1$ which can no be smaller than an arbitrarily small number $\epsilon$. Thus, $g_n(x)$ is not uniformly convergent.

**Definition 5.4.** *For a given series $\sum_{n=0}^{\infty} u_n(x)$, if the partial sum $\{S_N(x)\}$ is uniformly convergent in an interval $[a, b]$, then the series is called uniformly convergent in the interval $[a, b]$.*

It is not easy to check whether a series is a uniformly convergent or not according to the definition. How do we know if a series is uniformly convergent without using the partial sum and the definition? The idea is to compare the series with a convergent series that does not have $x$ in the series, which is always uniformly convergent. This is summarized in the Weierstrass M-test theorem.

**Theorem 5.5.** *Weierstrass M-test theorem. Given a series $\sum_{n=0}^{\infty} u_n(x)$ that satisfies* *the following conditions*

$$(i): \quad |u_n(x)| \le M_n \quad \textit{independent of } x \textit{ in an interval } [a,b], \tag{5.30}$$

$$(ii): \quad \sum_{n=0}^{\infty} M_n < \infty \quad \textit{the series that does not have } x \textit{ converges}, \tag{5.31}$$

*then the series is uniformly convergent in the interval* $[a,b]$.

**Example 5.18.** *Are the following series uniformly convergent? Find intervals that the series are uniformly convergent.*

$$(a): \quad \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}, \qquad (b): \quad \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad (c): \quad \sum_{n=1}^{\infty} e^{-nx} \sin nx.$$

**Solution:** (a): We know that

$$\left|\frac{\sin nx}{n^2}\right| \le \left|\frac{1}{n^2}\right| \quad \text{and the series} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent. Thus, the series is uniformly convergent. For (b), we can not use the Weierstrass M-test since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. So we do not have a conclusion about the uniform convergence since the theorem is a sufficient but not necessary. We will see the series cannot be uniformly convergent below. For (c), in any interval $(a,b)$ where $a > 0$, we have

$$|e^{-nx} \sin nx| \le e^{-nx} \le e^{-na}, \qquad \text{and} \quad \sum_{n=1}^{\infty} e^{-na}$$

is convergent. Thus, the series is uniformly convergent in $(a,b)$ when $a > 0$. The series does not converge if $x < 0$, and it is convergent but not uniformly in any interval $(0,b)$ for $b > 0$.

**Theorem 5.6.** *If a series*

$$f(x) = \sum_{n=1}^{\infty} u_n(x)$$

*is uniformly convergent in an interval* $(a,b)$, *then the series, after we take a limit, or integrate, or differentiate, term by term, is still convergent in the interval* $(a,b)$, *that is,*

$$\left(\lim_{x\to x_0};\ \frac{d}{dx};\ \int_\alpha^\beta dx\right) f(x) = \sum_{n=0}^{\infty} \left(\lim_{x\to x_0};\ \frac{d}{dx};\ \int_\alpha^\beta dx\right) u_n(x).$$

We may be able to use the theorem to further determine the uniform convergency of a series if the Weierstrass M-test fails. Reconsider the example (b) above for the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ for which the Weierstrass M-test fails. If the series was uniformly convergent, then we could take the derivative term by term to get a convergent series which is obviously untrue since the series $\sum_{n=1}^{\infty} \cos nx$ after the differentiation term by term diverges for almost any $x$. Thus, the series is not uniformly convergent. Another rule of thumb is that if the partial sums have Gibb's oscillations, then the series are not uniformly convergent. Therefore, the Fourier series for sawtooth function, the fractional part of $x$, and most half-range sine expansions are not uniformly convergent, while the Fourier series for triangular wave, and most half-range cosine expansions are uniformly convergent.

## 5.6 Exercises

**E5.1** Find the period of the following functions. $f(x) = 5$, $f(x) = \cos x$, $f(x) = \cos(\pi x)$, $f(x) = \cos(px)$, $f(x) = \cos^2 x$, $f(x) = \cos x \sin x$, $f(x) = \cos(x/2) + 3\sin(2x)$, $f(x) = \tan(mx) + e^{\sin(2x)}$, and $\left\{\cos \dfrac{n\pi x}{L}\right\}_{n=0}^{\infty}$.

**E5.2** Expand $f(x) = x$ and $g(x) = x^2$, $0 < x < 2$ as following series.

**(a)**. (Half range) sine series.

**(b)**. (Half range) cosine series.

**(c)**. Fourier series.

Check the pointwise and uniform convergence of all the series. Plot or sketch the series and the partial sums for large $N$.

**E5.3** Given $\dfrac{\partial^2 u}{\partial t^2} - 2\dfrac{\partial^2 u}{\partial x^2} = 0$. Find the general solution, and also do the following.

**(a)**. Solve the Cauchy problem $(-\infty < x < \infty)$ with $u(x,0) = |x|$, $|x| \leq 1$ and $u(x,0) = 0$ elsewhere; and $\dfrac{\partial u}{\partial t}(x,0) = 0$. Sketch the solution for $t = 5$.

**(b)**. Solve the Cauchy problem $(-\infty < x < \infty)$ with $u(x,0) = e^{-x}\cos x$, $\dfrac{\partial u}{\partial t}(x,0) = \sin x$.

**(c)**. Solve the boundary value problem $0 < x < 3$ with $u(0,t) = u(3,t) = 0$; $u(x,0) = \sin(6\pi x)$, $\dfrac{\partial u}{\partial t}(x,0) = \sin(24\pi x)$.

**(d).** Solve the boundary value problem $0 < x < 3$ with $u(0,t) = u(3,t) = 0$; $u(x,0) = x + 1$, $\dfrac{\partial u}{\partial t}(x,0) = \cos x$.

**E5.4** Let $f(x) = x$, $-\pi \leq x < \pi$ be a $2\pi$-periodic function.

**(a).** Sketch the function in the interval $[-3\pi,\ 3\pi]$.

**(b).** Find the Fourier Series Expansion for $f(x)$. Sketch the partial sums $S_N(x)$ for large $N$, and the Fourier series.

**(c).** Are the formulas

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin(nx)\,dx$$

the same? Why? Evaluate the second integration.

**(d).** Find the common period of $1, \cos(nx), \sin(nx)$, $n = 1, 2, \cdots$.

**E5.5** Expand $f(x) = \cos(2x)$ according to the following. **Check** whether the series are uniformly convergent or not; sketch the partial sums $S_N(x)$ with large $N$ and the series (mark Gibb's oscillations or round-ups if applies); and write down the Parseval's identity.

**(a).** The classical Fourier series in $(-\pi,\ \pi)$.

**(b).** The Fourier series in $(-1,\ 1)$.

**(c).** Half range sine series in $(0,\ \pi)$.

**(d).** Half range cosine series in $(0,\ 1)$.

**(e).** Half range cosine series in $(0,\ \pi)$.

**Note:** Pay attention to normal modes. It is okay to use Maple in some cases. If you do by hand, you need to evaluate the integrals.

**E5.6** Write down the Parseval's identity corresponding to the classical Fourier series of $f(x) = x^2$.

**E5.7** Find the Fourier series of the following $f(x)$ in $(-p,\ p)$, sketch $f(x)$, the partial sums $S_N(x)$ with large $N$, and the Fourier series in $(-p,\ p)$.

**(a).** $f(x) = -1, \quad -p \leq x < 0; \quad f(x) = 1, \quad 0 \leq x < p.$

**(b).** $f(x) = a\left(1 - (\frac{x}{p})^2\right)$, $-p \leq x < p$.

**(c).** $f(x) = \begin{cases} c & \text{if } |x| \leq d, \\ 0 & d < |x| < p, \end{cases}$ where $0 < d < p$.

**E5.8** Find the half-range **sine and cosine** expansions of function $f(x)$ in $(0,\ p)$, sketch $f(x)$, the partial sums $S_N(x)$ with large $N$, and the Fourier series in $(0,\ p)$

**(a)**. $f(x) = 1$, $p = 1$.

**(b)**. $f(x) = 1$, $p = 3\pi$.

**(c)**. $p = 2, f(x) = \begin{cases} 0 & \text{if } 0 < x \le 1, \\ x - 1 & 1 < x < 2. \end{cases}$

**(d)**. $f(x) = \sin(\pi x)\cos(\pi x)$, $p = 2\pi$.

**E5.9** **(a).** Use the Parseval's identity and the Fourier series expansion of $f(x) = x/2$ in $(-\pi, \pi)$,

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

to obtain $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

**(b).** From (a) to obtain that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24}$.

**(c).** Combine (a) and (b) to derive the identity $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

**E5.10** Compute $\displaystyle\int_{-\pi}^{\pi} f^2(x)dx$ using the Parseval's identity.

$$(\mathbf{a}) : f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}; \qquad (\mathbf{b}) : f(x) = 1 + \sum_{n=1}^{\infty} \left( \frac{\cos nx}{3^n} + \frac{\sin nx}{n} \right).$$

**Hint:** Use the geometric series and the table of Zeta functions, see

`https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function.`

**E5.11** Use the Weierstrass M-test to judge whether the following series are uniformly convergent or not.

**(a)**. $\displaystyle\sum_{n=1}^{\infty} \left( \frac{\cos nx}{n^2} + \frac{\sin nx}{n^3} \right)$.

**(b)**. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}, \quad |x| \le 10.$

**(c)**. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(x/n)}{n}$ for all $x$.

Which series are continuous, differentiable, and integrable, on which intervals?

**E5.12 Group work:** Design a wave filter that can only certain frequencies to pass, say $[a, b]$. Use this example to check

$$W(x) = 2.5\sin x + 4\cos(3x) + 10\cos(20(x+\pi)) + 0.02\sin(100 * x) + \epsilon rand(x).$$

In the above, $rand(x)$ is a random number generator. In the above function, which frequency is dominant? If $\epsilon = 0$, what is the Fourier series of $W(x)$? Let $\epsilon = 10^{-5}$, find the Fourier series of $W(x)$ then carry out the filtering.

# Chapter 6

# Series solutions of PDEs of boundary value problems

In this chapter, we continue to discuss the method of separation of variables for various boundary value problems of partial differential equation. We will see more relations of the solution with the Sturm-Liouville eigenvalue problems, orthogonal expansions, and various Fourier series.

## 6.1 One-dimensional wave equations

Recall one-dimensional wave equations,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{6.1}$$

where $c > 0$ is called a wave number in physics. We have already known how to solve the problem for various situations.

- The general solution is $u(x,t) = F(x-ct) + G(x+ct)$, where $F(x)$ and $G(x)$ are two arbitrarily differentiable functions, that is, no conditions are attached to the partial differential equation.

- Solution to a Cauchy problem, that is, given $u(x,0) = f(x)$; $\frac{\partial u}{\partial t}(x,0) = g(x)$, $\infty < x < \infty$, the solution is given by the D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2c}\int_{x-at}^{x+at} g(s)ds.$$

- Solution to some boundary value problems with normal modes initial conditions, that is, given an interval $0 < x < L$, the normal modes solution for some special $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, which can be expressed as

$$f(x) = \sum_{n=1}^{n} a_n \sin \frac{n\pi x}{L}, \qquad g(x) = \sum_{n=1}^{n} b_n \sin \frac{n\pi x}{L}.$$

The solution is

$$u(x,t) = \sum_{n=1}^{n} \left( a_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} + \frac{b_n}{n\pi c} \sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L} \right).$$

- General initial conditions, $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$, $0 < x < L$. The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( b_n \cos \frac{cn\pi t}{L} + b_n^* \sin \frac{cn\pi t}{L} \right),$$

where the coefficients are determined by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx,$$

$$b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx.$$

Note that the coefficients $b_n$'s are obtained from the half-range sine expansion of $f(x)$ and $b_n^*$'s are obtained from the half-range sine expansion of $g(x)$ by a constant that depends on $n$.

Now we discuss how to solve more general one-dimensional wave equations.

**Example 6.1.** *An example with non-homogeneous boundary condition.*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,$$

$$u(0,t) = u_0, \qquad u(L,t) = u_0,$$

$$u(0,t) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L.$$

In this case, we can use the transformation

$$v(x,t) = u(x,t) - u_0$$

to get the homogenous BC for $v(x,t)$

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}, \qquad 0 < x < L,$$

$$v(0,t) = 0, \qquad v(L,t) = 0,$$

$$v(0,t) = f(x) - u_0, \qquad \frac{\partial v}{\partial t}(x,0) = g(x), \qquad 0 < x < L.$$

The solution then will be

$$u(x,t) = u_0 + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(b_n \cos\frac{cn\pi t}{L} + b_n^* \sin\frac{cn\pi t}{L}\right)$$

where the coefficients are determined by

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \\ b_n^* &= \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx. \end{aligned}$$

**Challenge:** How about different boundary condition $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(L,t) = 0$. What are the normal mode solutions?

## 6.2 Series solution of 1D wave equations with derivative boundary conditions

An example with a Neumann boundary condition is given below,

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \\ &u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0, \\ &u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L, \end{aligned}$$

We will get a different Sturm-Liouville eigenvalue problem and a different expansion.

**Step 1:** Let $u(x,t) = T(t)X(x)$ and plug its partial derivatives into the original PDE so that we can separate variables. The homogeneous boundary conditions require $X(0) = X'(L) = 0$. Differentiating $u(x,t) = T(t)X(x)$ with respect to $t$ and $x$, respectively, we get

$$\begin{aligned} \frac{\partial u}{\partial t} &= T'(t)X(x), \quad \frac{\partial^2 u}{\partial t^2} = T''(t)X(x); \\ \frac{\partial u}{\partial x} &= T(t)X'(x), \quad \frac{\partial^2 u}{\partial x^2} = T(t)X''(x). \end{aligned}$$

The wave equation can be re-written as

$$T''(t)X(x) = c^2T(t)X''(x) \quad \Longrightarrow \quad \frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{6.2}$$

This is because in the last equality, the left hand side is a function of $t$ while the right hand side is a function of $x$, which is possible only both of them are a constant independent of $t$ and $x$. We can get eigenvalues either for $X(x)$ or $T(t)$. Since we know the boundary condition for $X(x)$, naturally we should solve

$$\frac{X''(x)}{X(x)} = -\lambda \quad \text{or} \quad X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X'(L) = 0 \tag{6.3}$$

first.

**Step 2:** Solve the eigenvalue problem. From the Sturm-Liouville eigenvalue theory, we know that $\lambda > 0$ since the $q(x)$ in the S-L eigenvalue problem is zero. Thus the solution of $x(x)$ is,

$$X''(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x.$$

From the boundary condition $X(0) = 0$, we get $C_1 = 0$. From the boundary condition $X'(L) = 0$, we get

$$C_2 \cos \sqrt{\lambda} L = 0, \quad \Longrightarrow \quad \sqrt{\lambda} L = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, \cdots,$$

since $C_2 \neq 0$. Note that, different from before, we should include $n = 0$. The eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = \left( \frac{(2n+1)\pi}{2L} \right)^2, \quad X_n(x) = \sin \frac{(2n+1)\pi x}{2L}, \quad n = 0, 1, 2, \cdots.$$

Now we solve for $T(t)$ using

$$T''(t) + c^2 \lambda_n T(t) = 0. \tag{6.4}$$

The solution (not an eigenvalue problem anymore since we have already known $\lambda_n$) of $T(t)$ is,

$$T_n(t) = b_n \cos \frac{(2n+1)\pi ct}{2L} + b_n^* \sin \frac{(2n+1)\pi ct}{2L}.$$

Putting $X_n(x)$ and $T_n(t)$ together, we get one normal mode solution with each $n$

$$u_n(x,t) = \sin \frac{(2n+1)\pi x}{2L} \left( b_n \cos \frac{(2n+1)\pi ct}{2L} + b_n^* \sin \frac{(2n+1)\pi ct}{2L} \right), \tag{6.5}$$

which satisfy the PDE, the boundary conditions, but not the initial conditions.

**Step 3:** Put all the normal mode solutions together to get the series solution using the superposition. The coefficients are obtained from the orthogonal expansion of the initial conditions. The solution to the IVP-BVP of the 1D wave equation with a derivative boundary condition can be written as

$$u(x,t) = \sum_{n=0}^{\infty} \sin \frac{(2n+1)\pi x}{2L} \left( b_n \cos \frac{(2n+1)\pi ct}{2L} + b_n^* \sin \frac{(2n+1)\pi ct}{2L} \right) \tag{6.6}$$

which satisfies the PDE and the boundary conditions. The coefficients of $b_n$ and $b_n^*$ are determined from the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$,

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin\frac{(2n+1)\pi x}{2L} \qquad \Longrightarrow \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n+1)\pi x}{2L}dx$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi x}{2L}\left(-b_n\frac{(2n+1)\pi c}{2L}\sin\frac{(2n+1)\pi ct}{2L}\right.$$
$$\left. + b_n^*\frac{(2n+1)\pi c}{2L}\cos\frac{(2n+1)\pi ct}{2L}\right),$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi x}{2L} b_n^*\frac{(2n+1)\pi c}{2L} \qquad \Longrightarrow$$

$$b_n^* = \frac{\dfrac{2L}{(2n+1)\pi c}\displaystyle\int_0^L g(x)\sin\frac{(2n+1)\pi x}{2L}dx}{\displaystyle\int_0^L\left(\sin\frac{(2n+1)\pi x}{2L}\right)^2 dx}$$
$$= \frac{4}{(2n+1)\pi c}\int_0^L g(x)\sin\frac{(2n+1)\pi x}{2L}dx.$$

### 6.2.1 Summary of series solutions of 1D wave equations with homogeneous linear BC's

From above discussions, we can summarize the series solutions to 1D wave equations with different boundary conditions below.

**Series solutions of 1D wave equations**

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x); \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 < x < L,$$

with homogeneous Dirichlet, Neumann, and Robin boundary conditions have the following uniform form.

$$u(x,t) = \sum_{n=1\ or\ 0}^{\infty} X_n\left(\frac{\alpha_n\pi x}{L}\right)\left(b_n\cos\frac{\alpha_n\pi ct}{L} + b_n^*\sin\frac{\alpha_n\pi ct}{L}\right) \tag{6.7}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)X_n\left(\frac{\alpha_n\pi x}{L}\right)dx, \quad b_n^* = \frac{2}{\alpha_n\pi c}\int_0^L g(x)X_n\left(\frac{\alpha_n\pi x}{L}\right)dx. \tag{6.8}$$

**Dirichlet-Dirichlet:** $u(0,t) = u(L,t) = 0$. We have $X_n\left(\frac{\alpha_n \pi x}{L}\right) = \sin\frac{n\pi x}{L}$.

**Dirichlet-Neumann:**

$$u(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0. \quad \text{We have} \quad X_n\left(\frac{\alpha_n \pi x}{L}\right) = \sin\frac{(\frac{1}{2}+n)\pi x}{L}.$$

**Neumann-Dirichlet:**

$$\frac{\partial u}{\partial x}(0,t) = 0,\ u(L,t) = 0. \quad \text{We have} \quad X_n\left(\frac{\alpha_n \pi x}{L}\right) = \cos\frac{(\frac{1}{2}+n)\pi x}{L}.$$

**Neumann-Neumann:** Any constants are solutions so the solution is not unique.

**Dirichlet-Robin:** $u(0,t) = 0,\ u(L,t) + \frac{\partial u}{\partial x}(L,t) = 0$. We have $X_n\left(\frac{\alpha_n \pi x}{L}\right) = \sin\frac{\alpha_n \pi x}{L}$, where we do not have analytic expressions for $\alpha_n$.

For Robin-Dirichlet, or Neumann-Robin, or other linear boundary conditions, we generally do not have an analytic form for the eigenvalues and eigenfunctions. Thus, there are no analytic series solutions available.

### 6.2.2 Series solution of 1D wave equations of BVPs with a lower order term

Consider an example of a 1D wave equation with a lower order term of the following,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + a^2 u = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\\
&u(0,t) = 0, \qquad u(L,t) = 0,\\
&u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).
\end{aligned}$$

**Solution:** The method of separation variables with $u(x,t) = T(t)X(x)$ will lead to

$$\frac{T''(t) + a^2 T(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{6.9}$$

We still have $\lambda_n = (\frac{n\pi}{L})^2$ and $X_n(x) = \sin\frac{n\pi x}{L}$. But the solution of $T_n(t)$ will be different.

$$T_n(t) = b_n \cos\left(\sqrt{a^2 + \frac{c^2 n^2 \pi^2}{L^2}}\, t\right) + b_n^* \sin\left(\sqrt{a^2 + \frac{c^2 n^2 \pi^2}{L^2}}\, t\right). \tag{6.10}$$

The solution then is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \left\{ b_n \cos\left(\sqrt{a^2 + \frac{c^2 n^2 \pi^2}{L^2}}\, t\right) + b_n^* \sin\left(\sqrt{a^2 + \frac{c^2 n^2 \pi^2}{L^2}}\, t\right)\right\},$$

with $b_n$ being the coefficient of the half range sine expansion of $f(x)$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx, \tag{6.11}$$

and

$$b_n^* = \frac{2}{L\alpha_n}\int_0^L g(x) \sin\frac{n\pi x}{L} dx, \quad \text{where} \quad \alpha_n = \sqrt{a^2 + \frac{c^2 n^2 \pi^2}{L^2}}. \tag{6.12}$$

**Challenge for a group work:** How about the modified PDE below

$$\frac{\partial^2 u}{\partial t^2} + au = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,$$

assuming $a$ is an arbitrary constant with the same boundary and initial conditions above? The solution for $T(t)$ may have two parts, the exponential functions and trigonometric functions.

## 6.3 Series solution to 1D heat equations with various BC's

A one-dimensional (1D) heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{6.13}$$

is a good mathematical model of the temperature distribution in a rod in which $c^2$ is called the heat conductivity. The above partial differential equation is a second order, constant coefficients, linear, and homogeneous one. The PDE is classified as a *parabolic PDE*. We can check that

$$u(x,t) = \frac{1}{\sqrt{4\pi c^2 t}}\, e^{-\frac{x^2}{4c^2 t}} \tag{6.14}$$

is a solution to the PDE. It is called a *fundamental solution to the heat equation* in the place of general solutions for advection and wave equations. The fundamental solution of the heat equation corresponds to an instant heat source at $(0,0)$. The heat will be felt anywhere anytime instantly. Thus, the solution to a heat equation is called global one meaning that a change at any place at any time will affect the solution everywhere and anytime after. This is in contrast to solutions to advection and wave equations.

The solution to the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \\ &u(x,0) = f(x), \end{aligned} \tag{6.15}$$

is the convolution of $f(x)$ and the fundamental solution,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{f(\xi)}{\sqrt{4c^2\pi t}}\, e^{-\dfrac{(x-\xi)^2}{4c^2 t}}\, d\xi. \tag{6.16}$$

Note that there is only one initial condition since the PDE involves only the first order derivative of the solution with time $t$.

Now we discuss the initial and boundary value problems for one-dimensional heat equation with various boundary conditions. Note that for homogeneous Dirichlet boundary condition $u(0,t) = 0$ and $u(L,t) = 0$, the derivation and the formula of the series solution have been given in Section 4.6. We first review one example here.

**Example 6.2.** *We can solve some 1D heat equations of boundary value problems with normal mode initial conditions as in the example below.*

$$\begin{aligned} &\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 3, \\ &u(0,t) = 0, \qquad u(3,t) = 0, \\ &u(x,0) = 5\sin(4\pi x) - 3\sin(8\pi x) + 2\sin(10\pi x). \end{aligned}$$

**Solution:** It is clear that we can use the normal mode solutions for this problem with $L = 3$, $c^2 = 2$. The initial condition can be written as

$$\begin{aligned} u(x,0) &= B_1 \sin\frac{m_1\pi x}{3} + B_2 \sin\frac{m_2\pi x}{3} + B_3 \sin\frac{m_3\pi x}{3} \\ &= 5\sin(4\pi x) - 3\sin(8\pi x) + 2\sin(10\pi x), \end{aligned}$$

which is possible if and only if $B_1 = 5$, $m_1 = 12$, $B_2 = -3$, $m_2 = 24$, and $B_3 = 2$, $m_3 = 30$. From the formula

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-c^2(\frac{n\pi}{L})^2 t},$$

we get the solution

$$u(x,t) = 5e^{-32\pi^2 t}\sin(4\pi x) - 3e^{-128\pi^2 t}\sin(8\pi x) + 2e^{-200\pi^2 t}\sin(10\pi x).$$

Now we consider the solution to the boundary and initial value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\\
&\frac{\partial u}{\partial x}(0,t) = 0, \qquad u(L,t) = 0,\\
&u(x,0) = f(x),
\end{aligned} \tag{6.17}$$

using the method of separation of variables. Note that a homogeneous Neumann boundary condition is prescribed at $x = 0$.

**Step 1:** Let $u(x,t) = T(t)X(x)$ and plug its partial derivatives into the original PDE so that we can separate variables. The homogeneous boundary conditions require $X'(0) = 0$ and $X(L) = 0$. Differentiating with $u(x,t) = T(t)X(x)$ with $t$ and $x$ respectively, we get

$$\frac{\partial u}{\partial t} = T'(t)X(x), \qquad \frac{\partial u}{\partial x} = T(t)X'(x), \quad \frac{\partial^2 u}{\partial x^2} = T(t)X''(x).$$

The heat equation can be re-written as

$$T'(t)X(x) = c^2 T(t)X''(x) \quad \Longrightarrow \quad \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{6.18}$$

This is because in the last equality, the left hand side is a function of $t$ while the right hand side is a function of $x$, which is possible only both of them are a constant independent of $t$ and $x$. We can solve the eigenvalue problems either for $X(x)$ or $T(t)$. Since we know the boundary condition for $X(x)$, naturally we should solve

$$\frac{X''(x)}{X(x)} = -\lambda \quad \text{or} \quad X''(x) + \lambda X(x) = 0, \quad X'(0) = X(L) = 0 \tag{6.19}$$

first.

**Step 2:** Solve the eigenvalue problem. From the Sturm-Liouville eigenvalue theory, we know that $\lambda > 0$ since the $q(x)$ term in the S-L theorem is zero. Thus, the solution of $X(x)$ is

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

$$X'(x) = -C_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + C_2 \sqrt{\lambda} \cos \sqrt{\lambda} x.$$

From the boundary condition $X'(0) = 0$, we get $C_2 = 0$. From the boundary condition $X(L) = 0$, we get

$$C_1 \cos \sqrt{\lambda} L = 0, \quad \Longrightarrow \quad \sqrt{\lambda} L = n\pi + \frac{\pi}{2}, \quad n = 0, 1, \cdots,$$

since $C_1 \neq 0$. The eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = \left( \frac{(2n+1)\pi}{2L} \right)^2, \quad X_n(x) = \cos \frac{(2n+1)\pi x}{2L}, \quad n = 0, 1, 2, \cdots.$$

Now we solve for $T(t)$ using

$$T'(t) + c^2 \lambda_n T(t) = 0. \tag{6.20}$$

The solution (not an eigenvalue problem anymore since we have already known $\lambda_n$) of $T(t)$ is

$$T_n(t) = a_n e^{-c^2 \lambda_n t} = b_n e^{-c^2 \left( \frac{(2n+1)\pi}{2L} \right)^2 t}.$$

Putting $X_n(x)$ and $T_n(t)$ together, we get a normal mode solution for each $n$,

$$u_n(x,t) = a_n \cos \frac{(2n+1)\pi x}{2L} e^{-c^2 \left( \frac{(2n+1)\pi}{2L} \right)^2 t}, \tag{6.21}$$

which satisfy the PDE, the boundary conditions, but not the initial condition.

**Step 3:** Put all the normal mode solutions together to get the series solution. The coefficients are obtained from the orthogonal expansion of the initial condition.

Thus, the solution to the initial and boundary value problem of the 1D wave equation can be written as

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos \frac{(2n+1)\pi x}{2L} e^{-c^2 \left( \frac{(2n+1)\pi}{2L} \right)^2 t} \tag{6.22}$$

which satisfies the PDE and the boundary conditions. The coefficients of $b_n$ are determined from the initial conditions $u(x,0)$,

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos \frac{(2n+1)\pi x}{2L} \qquad \Longrightarrow \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n+1)\pi x}{2L} dx,$$

which is a Fourier expansion of the initial condition $u(x,0) = f(x)$ on $(0, L)$.