Units and Dimensional Analysis

- Math modeling is to used to solve real world problems. Most of quantities in the real world have units. Or physical quantities are measured using units.
- A **unit of measurement** is a definite magnitude of a physical quantity, defined and adopted by convention or by law, that is used as a standard for measurement of the same physical quantity. Any other value of the physical quantity can be expressed as a simple multiple of the unit of measurement.

Dimensional Analysis

- Purposes:
  - Check the correctness of mathematical models for physical problems by checking dimensional unit
  - Derive mathematical models
  - Reduce parameters through ***Non-dimensionalization process or (scaling)***
  - Identify key parameters such as Reynolds number, peck number etc.

Units Examples

- Length is a physical quantity. The meter is a unit of length that represents a definite predetermined length. When we say 10 meters (or 10 m), we actually mean 10 times the definite pre-determined length called “meter (metre)".
- The definition, agreement, and practical use of units of measurement have played a crucial role in human endeavour from early ages up to this day. Different systems of units used to be very common. Now there is a global standard, the International System of Units (SI), the modern form of the metric system. But English system is still in use (UK, US, some counties in British Commonwealth Union)

Units

The recommended scientific system of units is the SI system, which includes 7 basic units.

Quantity	Unit	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Luminous intensity	candela	cd
Amount of substance	mole	mol

Large and small units

There is a wide range of sizes for all sorts of quantities. So we use special symbols for large and small multiples of the basic units.

Multiplication factor	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n

Commonly used units

There are also other commonly used units which are combinations of some of the SI units.

Quantity	Unit	Symbol
Force	newton	N ($kg\ m\ s^{-2}$)
Energy	joule	J ($kg\ m^2\ s^{-2}$)
Power	watt	W ($J\ s^{-1}$ or $kg\ m^2\ s^{-3}$)
Frequency	hertz	Hz (s^{-1})
Pressure	pascal	Pa ($N\ m^{-2}$ or $kg\ m^{-1}\ s^{-2}$)

Non-SI units

There are a number of non-SI units which are in common use by scientists and engineers.

Quantity	Unit	Symbol
Area	hectare	ha (= 10^4 m^2)
Volume	litre	l (= 10^{-3} m^3)
Volume	millilitre	ml (= 10^{-6} m^3)
Temperature	degree Celsius	°C (0°C ≈ 273 K)
Mass	gram	g (= 10^{-3} kg)
Mass	tonne	t (= 10^3 kg)
Energy	kilowatt hour	kW h (= 3.6×10^6 J)
Energy	electronvolt	eV (≈ 1.6×10^{-19} J)
Energy	calorie	cal (= 4.1868 J)
Pressure	bar	bar (= 10^5 Pa)
Pressure	atmosphere	atm (≈ 1.013×10^5 Pa)

Units Converting

The Fahrenheit scale of temperature

To convert from ^oF to ^oC,

$^oC = 5(^oF - 32)/9 \sim (^oF-32)/2$

To convert from ^oC to ^oF

$^oF = 9\,^oC/5 + 32 \sim 2\,^oC + 32$

Converting any units (almost anything)

http://www.onlineconversion.com/

British units system

Another system of units which is still in use is the British system.

fps unit	SI equivalent
inch (in)	0.0254 m
foot (ft)	0.3048 m
mile (5280 ft) (5 miles $\approx$ 8 km)	$1.609\,344 \times 10^3$ m
nautical mile (6080 ft)	$1.853\,184 \times 10^3$ m
acre	$4.046\,856 \times 10^3$ m^2($\approx$ 0.4 ha)
pint (pt)	$5.682\,613 \times 10^{-4}$ m^3
gallon (gal)	$4.546\,09 \times 10^{-3}$ m^3
ounce (oz)	$2.834\,952 \times 10^{-2}$ kg
pound (lb)	0.453 592 37 kg
horsepower (hp)	7.457×10^2 W

An example

The price of milk in the UK is about 1.65 pounds every 6 pints. That in China is 33 RMB every 6 litre. Assume that 1 pound = 15 RMB.

- ❑ Which is cheaper? (using the same unit)

  2008: UK: 1.65/6 → 15*1.65/3.409=7.26,   CH: 33/6=5.5

  2013: UK: 1.65/6 → 9.37*1.65/3.409=4.53,   CH: 33/6=5.5

- ❑ There is a 50% price rise in China recently. Which is cheaper? CH: 33*1.5/6=5.5*1.5=8.25

Dimensional Analysis

- A model which describes a physical, biological, economic or managerial system involves a variety of parameters or variables
- With each variable or parameter we can associate a dimension

area = length^2

velocity = length / time

Dimensions: Definition

- All ***mechanical*** quantities can be expressed in terms of the fundamental quantities

  mass (M) or kg, length (L) or m, time (T) or s

- Other physical quantities can be expressed as a combination of these 3 terms
- The resultant combination is called the '*dimensions*' of that physical quantity

Dimensions: Definition

- We use square brackets [ ] to denote "the dimension of "

$$[\text{area}] = L^2 \qquad [\text{speed}] = L\,T^{-1}$$

$$[\text{density}] = M\,L^{-3} \qquad [\text{angle}] = L\,L^{-1} = L^0$$

$$[\text{force}] = MLT^{-2} \qquad [\text{weight}] = M\,L\,T^{-2}$$

Note, dimensions are independent of the units used

Full Dimensional List

Mass	-	M
Length	-	L
Time	-	T
Electric Charge	-	Q
Temperature	-	θ
Number of Moles	-	MOL
Luminosity	-	?

Dimensional Analysis

- Any sensible equation must be dimensionally consistent

  [left-hand side] = [right-hand side]

- It is a good idea to carry out this check on all the equations appearing in a model

- This reveals any modeling errors

Dimensional Analysis

- Addition of terms only makes sense if each term has the same dimensions
- For a proposed equation, each term must be checked for consistency

  $A = B + (C \times D)$

- A, B and (C x D) must have the same dimensions

Determine the units for constants

Any constants appearing in equations can be

- Either be dimensionless (pure numbers)
- Or can have dimensions

Example

Suppose that we are modeling the force on a moving object due to air resistance. If we assume the magnitude of the force F is proportional to the square of the speed v:

$$F = kv^2$$

In dimensions: $[F] = [kv^2]$

$$MLT^{-2} = [k][LT^{-1}]^2 = [k]L^2T^{-2}$$

- For consistency, we require $[k] = ML^{-1}$
- k is measured in kg m^{-1}.

Dimensional Analysis

- If expressions involving ***exp(at)*** or ***sin(at)*** appear in our model, where t stands for time
- The parameter a must have dimensions T^{-1} so that ***at*** is a dimensionless number.
- If an equation involves a derivative, the dimensions of the derivative are given by the ratio of the dimensions
- If p is the pressure in a fluid at any point, z is the depth, then

Dimensional Analysis: example

$$\left[\frac{\mathrm{d}p}{\mathrm{d}z}\right] = \frac{[p]}{[z]} = \frac{\mathrm{ML^{-1}T^{-2}}}{\mathrm{L}} = \mathrm{ML^{-2}T^{-2}}$$

$$\left[\frac{\partial p}{\partial t}\right] = \frac{[p]}{[t]} = \frac{\mathrm{ML^{-1}T^{-2}}}{\mathrm{T}} = \mathrm{ML^{-2}T^{-3}}$$

$$\left[\frac{\partial^2 v}{\partial x^2}\right] = \frac{[v]}{[x]^2} = \frac{\mathrm{L\,T^{-1}}}{\mathrm{L^2}} = \mathrm{L^{-1}T^{-1}}$$

Dimensional Analysis: Pendulum

Suppose that we are trying to develop a model which will predict the period t of a swinging pendulum:-

List of Factors (attention restricted to 4 factors)

- the length l
- the mass m
- the angle θ
- acceleration g due to gravity

- Assume that the period

$$[t] = [k l^a m^b g^c \theta^d]$$

where a, b, c, d and k are unknown real numbers

Dimensional Analysis: Pendulum

Considering dimensions, we have

$$[t] = [kl^a m^b g^c \theta^d]$$

θ is dimensionless and *k* is assumed to be as well, so

$$\mathrm{T} = \mathrm{L}^a \mathrm{M}^b (\mathrm{LT}^{-2})^c$$

Equating powers of M, L and T on both sides

$$\left.\begin{aligned} \mathrm{L} &: 0 = a + c \\ \mathrm{M} &: 0 = b \\ \mathrm{T} &: 1 = -2c \end{aligned}\right\} \Rightarrow b = 0,\ \ c = -\frac{1}{2},\ \ a = -c = \frac{1}{2}$$

Dimensional Analysis: Pendulum

$$[t] = [kl^a m^b g^c \theta^d]$$

therefore

$$t = kl^{1/2} g^{-1/2} \theta^d$$

At present, *d* is unresolved, i.e., it can take any value

Summing terms of this form leads to the general result

$$t = f(\theta) l^{1/2} g^{-1/2}$$

Dimensional Analysis: Fluid

Example: The pressure p at a depth h below the surface of a fluid of density ρ is given by p = ρgh, where g is the acceleration due to gravity. Check the dimensions.

$$[p] = \left[\frac{\text{force}}{\text{area}}\right] = \frac{\text{MLT}^{-2}}{\text{L}^2} = \text{ML}^{-1}\text{T}^{-2}$$

$$[\rho] = \text{ML}^{-3}$$

$$[g] = [\text{acceleration}] = \text{LT}^{-2}$$

$$[h] = \text{L}$$

$$[\rho g h] = \text{ML}^{-3}\text{LT}^{-2}\text{L} = \text{ML}^{-1}\text{T}^{-2}$$

The dimensions are consistent.

Nondimensionalization

- **Nondimensionalization** is the partial or full removal of units from an equation involving physical quantities by a suitable substitution of variables. This technique can simplify and parameterize problems where measured units are involved. It is closely related to dimensional analysis. In some physical systems, the term **scaling** is used interchangeably with *nondimensionalization*, in order to suggest that certain quantities are better measured relative to some appropriate unit. These units refer to quantities intrinsic to the system, rather than units such as SI units.
- Nondimensionalization can also recover characteristic properties of a system. The technique is especially useful for systems that can be described by differential equations.
- Nondimensionalization can reduce the number of parameters and keep the most important ones such as Reynolds number.

Nondimensionalization steps

To nondimensionalize a system of equations, one must do the following:

1. Identify all the independent and dependent variables;
2. Replace each of them with a quantity scaled relative to a characteristic unit of measure to be determined;
3. Divide through by the coefficient of the highest order polynomial or derivative term;
4. Choose judiciously the definition of the characteristic unit for each variable so that the coefficients of as many terms as possible become 1;
5. Rewrite the system of equations in terms of their new dimensionless quantities.

An illustrative example

$$a\frac{dx}{dt} + bx = Af(t).$$

1. In this equation the independent variable here is *t*, and the dependent variable is *x*.
2. Set $x = \chi x_c,\ t = \tau t_c$. This results in the equation

$$a\frac{x_c}{t_c}\frac{d\chi}{d\tau} + bx_c\chi = Af(\tau t_c) \stackrel{\text{def}}{=} AF(\tau).$$

3. The coefficient of the highest ordered term is in front of the first derivative term. Dividing by this gives

$$\frac{d\chi}{d\tau} + \frac{bt_c}{a}\chi = \frac{At_c}{ax_c}F(\tau).$$

4. The coefficient in front of χ only contains one characteristic variable t_c, hence it is easiest to choose to set this to unity first:

$$\frac{bt_c}{a} = 1 \Rightarrow t_c = \frac{a}{b}.$$ Subsequently, $$\frac{At_c}{ax_c} = \frac{A}{bx_c} = 1 \Rightarrow x_c = \frac{A}{b}.$$

5. The final dimensionless equation in this case becomes completely independent of any parameters with units:

$$\frac{d\chi}{d\tau} + \chi = F(\tau).$$

Dimensional Analysis

- Well known to any engineer
- This simple information is also remarkably useful in mathematical modelling
- Will consider ideas in context of the modelling of physical systems, but are easily extended to any application

Dimensional Analysis

- Check the validity of equations proposed during the modelling process
- Find a number of independent parameter groups (and calculate them)
- Find the relative sizes of various terms when model equations are proposed
- Normalise problems in terms of non-dimensional variables whose typical scale is of the order of one, and hence simplify them