Sample Final

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1. Given four points in an O-XYZ coordinates:

$$\text{A}\,(0,1,0), \qquad \text{B}\,(0,-1,1), \qquad \text{C}\,(1,1,-1), \qquad \text{D}\,(2,0,-1).$$

   (a) Find the distance between B and D.

   (b) Find the vectors $\vec{BA}$ and $\vec{BC}$.

   (c) Find the dot product $\vec{BA}\cdot\vec{BC}$ and the angle between the vectors $\vec{BA}$ and $\vec{BC}$.

   (d) Find the cross product $\vec{BA}\times\vec{BC}$ and the area of the parallelogram formed by the vectors $\vec{BA}$ and $\vec{BC}$. Are A, B, and C in a straight line?

   (e) Find the equation of the sphere that centered at B with the radius 3.

   (f) Find the equation of line passing through B and C in parametric form, symmetric form, and the equation(s) (without parameter).

   (g) Are A, B, C, D in a same plane? (use triple product.)

   (h) Find the equation of the plane passing through A, B, and C. Find the distance between D and the plane.

2. Give a geometrical or physical meening of the following integrals:

$$\int_C ds, \qquad \iint_D dxdy, \qquad \iint_S ds, \qquad \int_C \vec{F}\cdot dr, \qquad \iiint_V dxdydz,$$
$$\iiint_V \sigma dxdydz, \qquad \iint_S \vec{V}\cdot d\vec{S}.$$

   where C is a curve, S is a surface, D is a domain on xy plane, V is a solid in xyz space, $\vec{F}$ is a force, σ is a density function, $\vec{V}$ is a velocity function.

3. Given $\mathbf{r}(t) = (\cos t,\ \ \sin t,\ \ t)$, $0 \le t \le 4\pi$.

   (a) Find the tangent direction at $(0,1,\pi/2)$. Find the equation of the tangent plane at this point.

   (b) Find the length of the curve between 0 and 2π.

4. Give examples, equations, of the following surfaces. Ellipsoid, elliptic paraboloid, hyperbolic hyperboloid, cones, spheres, half planes.

5. Given $f(x,y) = \dfrac{xy}{x^2+y^2}$, does $f(x,y)$ has limit at $(-1,1)$? at $(0,0)$? Is $f(x,y)$ continuous?

6. Given

$$f(x,y) = \sqrt{9-x^2-y^2}$$

(a) Find the domain and range of $f(x,y)$. Is $f(x,y)$ continuous on the domain? Explain the definition of the continuity.

(b) Sketch the level curves of $f(x,y)$, i.e. $f(x,y)=k$, for $k=,0,1,2,3$.

(c) Sketch the graph of $f(x,y)$.

(d) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y^2}$. Under what kind of condition(s), we can conclude that $\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial y \partial x}$?

(e) Find the equation of the tangent plane of the graph $z=f(x,y)$ at $x=1$ and $y=1$.

(f) Given $\vec{u}=-\vec{i}+2\vec{j}$, find the directional derivative of $f(x,y)$ along the direction. Does $f(x,y)$ increase or decrease along this direction?

(g) In what direction does f has the maximum rate of increase and decline?

7. If $z=x/y$, $x=re^{st}$, $y=rse^{st}$. Evaluate the following:

$$\frac{\partial z}{\partial r}, \quad \frac{\partial z}{\partial s}, \quad \frac{\partial z}{\partial t}, \quad \frac{\partial z^2}{\partial r^2}{}^*.$$

8. (a) Find all critical points of the function

$$f(x,y)=2x^4-x^2+3y^2+4,$$

and then use the second derivative test to determine if each critical point corresponds to a local extreme value or saddle point of the function.

(b) Find the absolute maximum and minimum in the domain bounded by $x=0$, $y=0$, and $x+2y=2$.

(c) Find the equation of the tangent plane of the graph at $(0,0,4)$.

9. (a) Find y' if $x^3+y^3=6xy$.

(b) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$.

10. Evaluate the following integrals.

(a) $\iint_D f(x,y)\,dxdy$, where $f(x,y)=xy$, $D=\{\ (x,y), \quad 0\le x\le 2, \quad 0\le y\le 4\ \}$.

(b) $\int_0^1\int_0^z\int_0^y 2x\,dx\,dy\,dz$.

(c) $\iiint_V f(x,y,z)\,dxdydz$, where $V=\{\,0\le y\le 1, \quad 0\le z\le y, \quad 0\le x\le y-z\}$. Write it as an iterated integral of **six** different forms.

11. Given

$$\int_0^2 dx \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}}(x^2+y^2)dy.$$

(a) Change the order of integration, that is, integrate with x first.

(b) Express and evaluate the integral using the poplar coordinates. Which approach is the easiest for this problem?

12. Express the double integrals $\iint_D f(x, y)$ as two different iterated integrals both in Cartesian coordinates and in polar coordinates. Using the one which you think is simpler to evaluate the integrals. $f(x, y) = x \cos y$, D is bounded by $y = 0$, $y = 0$, and $x^2 + 2x + y^2 = 0$.

13. Problem of Sample Quiz 1-4, Quiz 1-4.

14. If the density of the laminar $x^2 + 2x + y^2 \leq 0$ is $\rho(x, y) = \sqrt{x^2 + y^2}$, find the mass, the center*, and the inertia* about the origin, of the laminar.

15. Find the triple integral $\iiint_V x\, dV$, where V is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$. Find the volume of V.

16. Express a triple integral as six different iterated integrals as in the Sample Quiz 4 and in Quiz 4.

17. Write down the spherical coordinates transformation and use it to evaluate the triple integral

$$\iiint\limits_V xyz(x^2 + y^2 + z^2) dx dy dz,$$

where V is the solid between two spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 4$, that lies in the half space first octant, *i.e.*, $x \geq 0$, $y \geq 0$, and $z \geq 0$.

18. Let $f(x, y, z) = x (\sin y) e^z$, $\vec{F} = (\sin x, \cos x, z^2)$.

(a) Find the gradient of $f(x, y, z)$, ∇f. What is $\nabla \times \nabla f$?

(b) Find the divergence of $\vec{F}$, $div\vec{F}$. Is $\vec{F}$ divergence free?

(c) Find the curl of $\vec{F}$, $curl\vec{F}$. Is $\vec{F}$ irrotational? Is $\vec{F}$ conservative?

(d) Evaluate the line integral $\int_C \nabla f \cdot d\vec{r}$, where C is the line segment from $(0, 1, 0)$ to $(1, \frac{\pi}{2}, 0)$.

19. Find the line integral $\int_C y ds$, $\int_{-C} y ds$, $\int_C x dx$, and $\int_{-C} x dx$, where C is the circle $x^2 + y^2 = 1$ in clockwise direction from $(0, 1)$ to $(1, 0)$ and the line segment from $(1, 0)$ to $(2, 3)$, and $-C$ is the same curve but in opposite direction.

20. (a) Use the Green's theorem to evaluate the line integral

$$\frac{1}{2} \oint_c (-y dx + x dy)$$

where c is the unit circle along the positive direction. What is the geometric meaning of your result?

(b)

$$\oint_c (yx^2 dx + x^2 y dy)$$

21. Given $\vec{F} = (y^2z^3, 2xyz^3, 3xy^2z^2)$.

    (a) Is $\vec{F}$ conservative? Why?

    (b) If your answer is yes, find a potential function $f(x, y, z)$ such that $\nabla f = \vec{F}$.

    (c) Evaluate the line integral $\int_c \vec{F} \cdot \vec{dr}$, where c is the helix $(4\cos t, 2\sin t, t)$ from $t = 4\pi$ to $t = \pi/2$.

22. (a) Give a parametric form of the surface

    $$x^2 + y^2 + z^2 = 4, \quad y \geq 0,$$

    determine the range of the parameters.

    (b) Find the unit normal direction of the surface and the differential area.

    (c) Find the equation of the tangent plane at $(0, 2, 0)$.

    (d) Find the area of the surface.

    (e) Find the intergral $\int\int_S f(x, y, z) ds$, and $\int\int_{-S} f(x, y, z) ds$.

    (f) Given a vector field $\vec{F} = (2x, -yz, xy)$, find the flux of $\vec{F}$, $\int\int_S \vec{F} \cdot \vec{ds}$ and $\int\int_{-S} \vec{F} \cdot \vec{ds}$ across the surface. where $-S$ is the same surface but with opposite normal direction.

23. Let $\vec{F} = (3y^2z^3, 9x^6yz^2, -4xy^2)$.

    (a) Find the divergence of $\vec{F}$.

    (b) Evaluate the flux of $\vec{F}$ across the cube: $-2 \leq x \leq 3$, $0 \leq y \leq 1$, $-3 \leq z \leq -1$.

24. Problems similar to Quizzes, class practices, homework, and examples explained in class.