Solution to Sample Quiz 4

1. $C_1 \begin{cases} x = \cos t \\ y = \sin t, \end{cases} \quad 0 \le t \le \frac{\pi}{2}$ $\quad \begin{matrix} dx = -\sin t\, dt \\ dy = \cos t\, dt \end{matrix} \quad ds = dt$

$C_2 \quad \begin{cases} x = 0 \\ y = y, \end{cases} \quad 1 \le y \le 2, \quad \begin{matrix} dx = 0 \\ dy = dy. \end{matrix} \quad ds = dy$

$$\int_C x^2 ds = \int_0^{\frac{\pi}{2}} \cos^2 t\, dt + \int_1^2 0 \cdot dy = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} dt$$

$$= \frac{\pi}{4} + \frac{1}{4} \sin 2t \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_C y\, dx = \int_0^{\frac{\pi}{2}} \sin t\,(-\sin t)\, dt + \int_1^2 y \cdot 0 = -\int_0^{\frac{\pi}{2}} \sin^2 t\, dt$$

$$= -\int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2} dt = -\frac{\pi}{4}$$

$$\int_{-C} y\, dx = -\int_C y\, dx = \frac{\pi}{4}$$

$$\text{Total length} = \int_{C_1} ds + \int_{C_2} ds = \int_0^{\frac{\pi}{2}} dt + \int_1^2 dy = \frac{\pi}{2} + 1$$

2. (a). $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x \sin y) + \frac{\partial}{\partial y}(y \cos y) + \frac{\partial}{\partial z}(xy)$

$$= \sin y + \cos y - y \sin y \ne 0.$$

$\vec{F}$ is not incompressible.

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\sin y & y\cos y & xy \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(y\cos y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(x\sin y)\right)$$
$$+ \vec{k}\left(\frac{\partial}{\partial x}(y\cos y) - \frac{\partial}{\partial y}(x \sin y)\right)$$

$$= \vec{i}(x-0) - \vec{j}(y-0) + \vec{k}(0 - x\cos y)$$

$$= (x, -y, -x\cos y) \ne \vec{0}$$ not conservative.

not irrotational.

The line integral will depend on path.

(b). $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(ye^{xy} + 4x^3y) + \frac{\partial}{\partial y}(xe^{xy} + x^4) + \frac{\partial}{\partial z}\cdot 0$

$= y^2e^{xy} + 12x^2y + x^2e^{xy} + 0 + 0 \neq 0$

$\vec{F}$ is not incompressible, not divergence free.

$$\operatorname{Curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy} + 4x^3y & xe^{xy} + x^4 & 0 \end{vmatrix}$$

Note: $\frac{\partial}{\partial z}(ye^{xy} + 4x^3y^3) = $

$\frac{\partial}{\partial z}(xe^{xy} + x^4) = 0$

$= \vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}\left(\frac{\partial}{\partial x}(xe^{xy} + x^4) - \frac{\partial}{\partial y}(ye^{xy} + 4x^3y)\right)$

$= \vec{k}\left(e^{xy} + xy\,e^{xy} + 4x^3 - (e^{xy} + xy\,e^{xy} + 4x^3)\right) = \vec{0}$

$\vec{F}$ is irrotational, conservative.

$P(x,y) = ye^{xy} + 4x^3y$, $\quad Q(x,y) = xe^{xy} + x^4$.

Let $f(x,y)$ is the potential function $\nabla f = \vec{F}$

$f(x,y) = \int P\,dx + c(y) = \int (ye^{xy} + 4x^3y)\,dx + c(y)$

$= e^{xy} + x^4y + c(y)$

$\frac{\partial f}{\partial y} = xe^{xy} + x^4 + c'(y) = Q(x,y)$

$= xe^{xy} + x^4$

$\Rightarrow c'(y) = 0. \Rightarrow \quad c(y) = K$ a constant.

$\Rightarrow f(x,y) = e^{xy} + x^4y + K$.

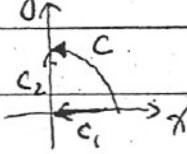
verify: $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$.

3. $P(x,y) = 2x\sin y$, $\quad Q(x,y) = x^2\cos y - 3y^2$

$\frac{\partial P}{\partial y} = 2x\cos y$, $\quad \frac{\partial Q}{\partial x} = 2x\cos y \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Therefore, the line integral is independent of path

Method 1 From $(2,0)$ to $(0,0)$



C_1: $\begin{cases} x = 2-2t \\ y = 0 \end{cases}$, $0 \le t \le 1$, $dx = -2\,dt$, $dy = 0$

Since $dy = 0$ and $\sin y = 0$ along C_1, the contribution of the line integral $\int_{C_1} \vec{F} \cdot d\vec{r} = 0$

From $(0,0)$ to $(0,3)$

C_2: $\begin{cases} x = 0 \\ y = 0+3t, \end{cases}$ $0 \le t \le 1$, $dx = 0$, $dy = 3\,dt$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 -3(3t)^2 \cdot 3\,dt = -27t^3\Big|_0^1 = -27.$$

Method 2. Find the potential function $\nabla f = \vec{F}$

i.e. $\frac{\partial f}{\partial x} = P(x,y) = 2x\sin y$, $f(x,y) = \int 2x \sin y\,dx + C(y)$

$f(x,y) = x^2 \sin y + C(y)$, Now use the second condition. $\frac{\partial f}{\partial y} = Q(x,y)$, $x^2\cos y + C'(y) = x^2\cos y - 3y^2$

Solve for $C'(y)$

$C'(y) = -3y^2$, $C(y) = \int -3y^2\,dy + K$

$C(y) = -y^3 + K$

Therefore $f(x,y) = x^2 \sin y - y^3 + K$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(0,3) - f(2,0)$$

$$= 0^2 \sin 3 - 3^3 - 2^2 \sin 0 - 0^3 = -27$$

We should get the same answer.

4. $P(x,y) = x^3 - y^3$, $Q(x,y) = x^3 + y^3$, $\frac{\partial P}{\partial y} = -3y^2$, $\frac{\partial Q}{\partial x} = 3x^2$

$\oint \vec{F} \cdot d\vec{r} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint 3(x^2+y^2)\,dx\,dy$

$= \int_0^{2\pi} d\theta \int_{r=1}^{r=3} 3r^2 \cdot r\,dr = 6\pi \left.\frac{r^4}{4}\right|_1^3 = \frac{6\pi}{4} \cdot 80 = 120\pi$

(5). $\iiint_V z^3 \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$

$= \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi \int_0^1 (\rho^3 \cos^3\varphi) \cdot \rho\, (\rho^2 \sin\varphi)\, d\rho$

$= 2\pi \left(\int_0^{\pi/2} \cos^3\varphi \sin\varphi\, d\varphi\right) \cdot \left(\int_0^1 \rho^6\, d\rho\right) = 2\pi \left[-\frac{1}{4}\cos^4\varphi\right]\Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{7} = \frac{\pi}{14}$