Solution to Sample Quiz 3.

1. (a) $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^x yz\,dy\,dz\,dx = \int_0^3\int_0^{\sqrt{9-x^2}} \frac{1}{2}x^2 z\,dz\,dx$

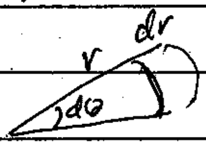
$= \int_0^3 \frac{1}{4}x^2 z^2\Big|_{z=0}^{z=\sqrt{9-x^2}} dx = \int_0^3 \frac{1}{4}(9x^2-x^4)\,dx$

$= \frac{1}{4}\left[3x^3 - \frac{x^5}{5}\right]\Big|_0^3 = \frac{1}{4}\cdot\frac{162}{5} = \frac{81}{10}$

(b) $\iint_D dx\,dy$ = The area of the domain D.

$\iiint_V dx\,dy\,dz$ = The volume of the domain V.

(c) $d\theta$ is the angle not the length.



d Area $\approx dr \cdot r\,d\theta$.

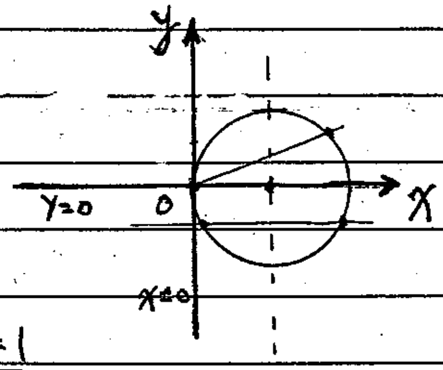
(d) After we integrate with z, we get function of x and y, after we integrate with y, we still have z in it, so after the integration, the result is a <u>function of z</u>.

(e) $\iiint_E dv = \int_0^1\int_{x=0}^{x=y}\int_{z=0}^{z=x+y} dz\,dx\,dy = \int_0^1\int_0^y (x+y)\,dx\,dy$

$= \int_0^1 \left(\frac{x^2}{2}+xy\right)\Big|_0^y dy = \int_0^1 \left(\frac{y^2}{2}+y^2\right)dy = \int_0^1 \frac{3}{2}y^2\,dy$

$= \frac{1}{2}y^3\Big|_0^1 = \frac{1}{2}$

2 (a) $0 \le x \le 2$, $\quad -\sqrt{2x-x^2} \le y \le \sqrt{2x-x^2}$



$y = \pm\sqrt{2x-x^2}$.

$\Rightarrow y^2 = 2x - x^2, \Rightarrow x^2+y^2 = 2x$

$y^2+x^2-2x=0 \Rightarrow \underline{y^2+(x-1)^2=1}$

Solve for x $\quad (x-1)^2 = 1-y^2 \qquad x = 1 \pm \sqrt{1-y^2}$

$$\int_0^2 dx \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (x^2+y^2)\,dy = \int_{-1}^{1} dy \int_{x=1-\sqrt{1-y^2}}^{x=1+\sqrt{1-y^2}} (x^2+y^2)\,dx$$

(b) In polar coordinates $x^2+y^2=2x$ becomes

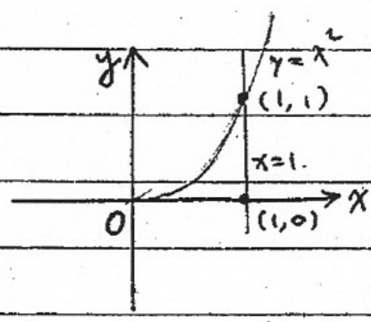
$$r^2 = 2r\cos\theta, \Rightarrow r=0$$
$$\text{or } r = 2\cos\theta$$

Therefore $\int_0^2 dx \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (x^2+y^2)\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{r=2\cos\theta} r^3\,dr\,d\theta$

polar coordinates is the easiest way to evaluate it.

3. (a) $\iint_D x\cos y\,dx\,dy = \int_0^1 dx \int_{y=0}^{y=x^2} x\cos y\,dy$

$$= \int_0^1 dy \int_{x=\sqrt{y}}^{x=1} x\cos y\,dx$$

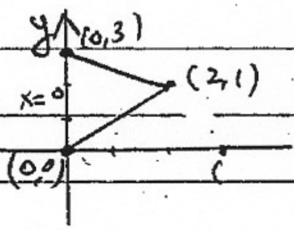


Integrate with y first.

$$\int_0^1 dx \int_0^{x^2} x\cos y\,dy = \int_0^1 x \sin y\Big|_0^{x^2} dx = \int_0^1 x\sin x^2\,dx$$

$$= -\frac{1}{2}\cos x^2\Big|_0^1 = \frac{1}{2}(1-\cos 1)$$

(b)



First we need to find the equations of the lines using the symmetric form

The Line eqn. between (0, 3) and (2, 1)

$$\frac{x-0}{2} = \frac{y-3}{-2}, \quad \text{or } y = 3-x.$$

The line eqn. between (2, 1) and (0, 0) is $y=\frac{1}{2}x$.

Integrating with x first requires to break the domain

So we integrate with y first.

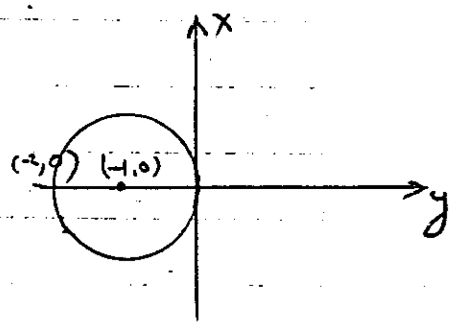
$$\iint_D y e^x \, dx\,dy = \int_0^2 \int_{y=\frac{x}{2}}^{y=3-x} y\, e^x \, dy\, dx$$

$$= \int_0^2 e^x \left. \frac{y^2}{2} \right|_{x/2}^{3-x} dx$$

$$= \int_0^2 \frac{1}{2} e^x \left((3-x)^2 - \frac{x^2}{4} \right) dx = \frac{9}{4} e^2 - \frac{33}{4}$$

4. $x^2 + 2x + y^2 = 0 \Rightarrow (x+1)^2 + y^2 = 1$.

$M = \iint_D \rho(x,y)\, dx\,dy$



In polar coordinates:

$$(x+1)^2 + y^2 - 1 = (r\cos\theta + 1)^2 + (r\sin\theta)^2 - 1$$

$$= r^2\cos^2\theta + 2r\cos\theta + r^2\sin^2\theta$$

$$= r^2 + 2r\cos\theta = r(r + 2\cos\theta) = 0$$

The boundary in polar coordinates is $r = -2\cos\theta$

$+\frac{\pi}{2} \le \theta \le \frac{3}{2}\pi$

$$M = \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} d\theta \int_{r=0}^{r=-2\cos\theta} r \cdot r \, dr$$

$$= \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} d\theta \left. \frac{r^3}{3} \right|_0^{-2\cos\theta} = \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} -\frac{8}{3}\cos^3\theta \, d\theta$$

$$= -\frac{8}{3} \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \cos\theta \, (1 - \sin^2\theta)\, d\theta$$

$$= -\frac{8}{3} \left[\sin\theta \Big|_{\frac{\pi}{2}}^{\frac{3}{2}\pi} - \frac{\sin^3\theta}{3} \Big|_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \right]$$

$$= -\frac{8}{3} \left[-2 + \frac{2}{3} \right] = \frac{32}{9}$$

5 $$V = \iiint dx\,dy\,dz = \iint_{D(x,y)} \int_{z=0}^{z=\sqrt{1-x^2-y^2}} dz\,dx\,dy$$

$$= \iint_{D(x,y)} \sqrt{1-x^2-y^2}\,dx\,dy$$

To find the domain, the maximum projection of the solid on the xy-plane, we note that the maximum (the tallest part) is the intersection of the two surfaces

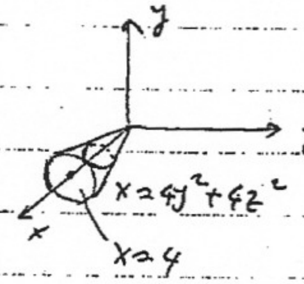
$$\begin{cases} x^2+y^2+z^2=1 \\ z^2 = x^2+y^2 \end{cases} \Rightarrow x^2+y^2 + \underbrace{x^2+y^2}_{z^2} = 1$$

or $x^2+y^2 = \left(\frac{1}{\sqrt{2}}\right)^2$ therefore

$$V = \int_0^{2\pi} \int_{r=0}^{r=\frac{1}{\sqrt{2}}} \sqrt{1-r^2}\; r\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{2}\,\frac{(1-r^2)^{3/2}}{\frac{1}{2}+1}\Bigg|_0^{\frac{1}{\sqrt{2}}} d\theta$$

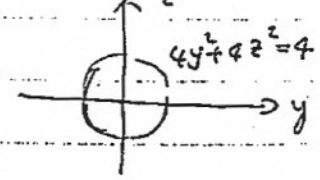
$$= -2\pi \cdot \frac{1}{3}\left[\left(1-\frac{1}{2}\right)^{3/2} - 1\right] = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{4}\right)$$

(6) Use the cylindrical coordinates.



$$\iiint_V x\,dV = \iint_D dy\,dz \int_{x=4y^2+4z^2}^{x=4} x\,dx$$

$$= \frac{1}{2} \iint_D \left(4^2 - (4y^2+4z^2)^2\right) dy\,dz$$



$\begin{cases} y = r\cos\theta \\ z = r\sin\theta \end{cases}$

$$= 8 \int_0^{2\pi} d\theta \int_0^1 (1-r^4)\,r\,dr$$

$$= 16\pi \int_0^1 (r - r^5)\,dr$$

$$= 16\pi \left[\left(\frac{r^2}{2} - \frac{r^6}{6}\right)\Bigg|_0^1 \right)$$

$$= \frac{16\pi}{3}$$

7. $$\iiint_V f\,dV = \int_{z=0}^{z=2} \int_{x=0}^{x=\frac{2-z}{2}} \int_{y=z-2x}^{y=2} f(x,y,z)\,dy\,dx\,dz$$