Solution to Sample Quiz 2

1\. (a) $\lim_{t\to\pi} \vec{r}(t) = \langle -3, 0, 0 \rangle$

(b) $\int \vec{r}(t)\,dt = \int 3\cos t\,\vec{i}\,dt - \int t\sin t\,\vec{k}\,dt$

$$= (3\sin t + C_1)\vec{i} + C_2\vec{j}$$
$$- (\sin t - t\cos t + C_3)\vec{k}$$

(c) $\vec{r}\,'(t) = \langle -3\sin t,\ 0,\ -\sin t - t\cos t \rangle$

$$L = \int_0^s |\vec{r}\,'(\mu)|\,d\mu = \int_0^s \sqrt{9\sin^2\mu + (\sin\mu - \mu\cos\mu)^2}\,d\mu$$

(d) $(3, 0, 0) = (3\cos t,\ 0,\ t\sin t)$

$3\cos t = 3, \qquad \cos t = 1, \qquad t = 0, 2\pi \cdots$

Take $t = 0$. $\vec{r}\,'(0) = \langle 0, 0, 0 \rangle$

The equation of the tangent line is

$$\begin{cases} x = 3 \\ y = 0 \\ z = 0 \end{cases}$$

2\. (a) $\lim_{(x,y,z)\to(1,\frac{\pi}{2},0)} \dfrac{x^2 \sin y\,(z-1)^3}{x+y-e^z} = \dfrac{1^2 \sin\frac{\pi}{2}\,(0-1)^3}{1+\frac{\pi}{2}-e^0} = -\dfrac{2}{\pi}$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x-y} = \lim_{(x,y)\to(0,0)} \dfrac{\cancel{(x-y)}(x^2+xy+y^2)}{\cancel{(x-y)}} = 0$

(c) Take $y = kx$

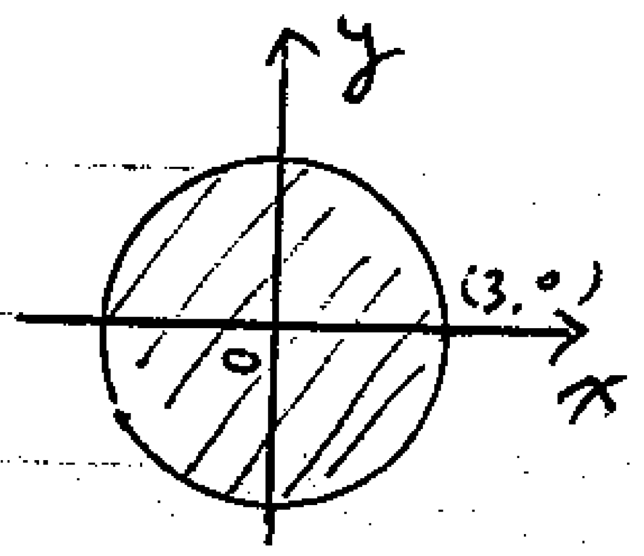
$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{x\to 0} \frac{x-kx}{x+kx} = \frac{1-k}{1+k} = \begin{cases} 0 & \text{if } k=1 \\ 1 & \text{if } k=0 \end{cases}$$

The limit does not exist.

(d) $\lim_{(x,y)\to(\infty,\infty)} \dfrac{\sin xy}{x^2+y^2} = 0$, since $0 \le \left|\dfrac{\sin xy}{x^2+y^2}\right| \le \dfrac{1}{x^2+y^2}$

3 (a) Domain $9 - x^2 - y^2 \geq 0$ or $x^2 + y^2 \leq 3^2$

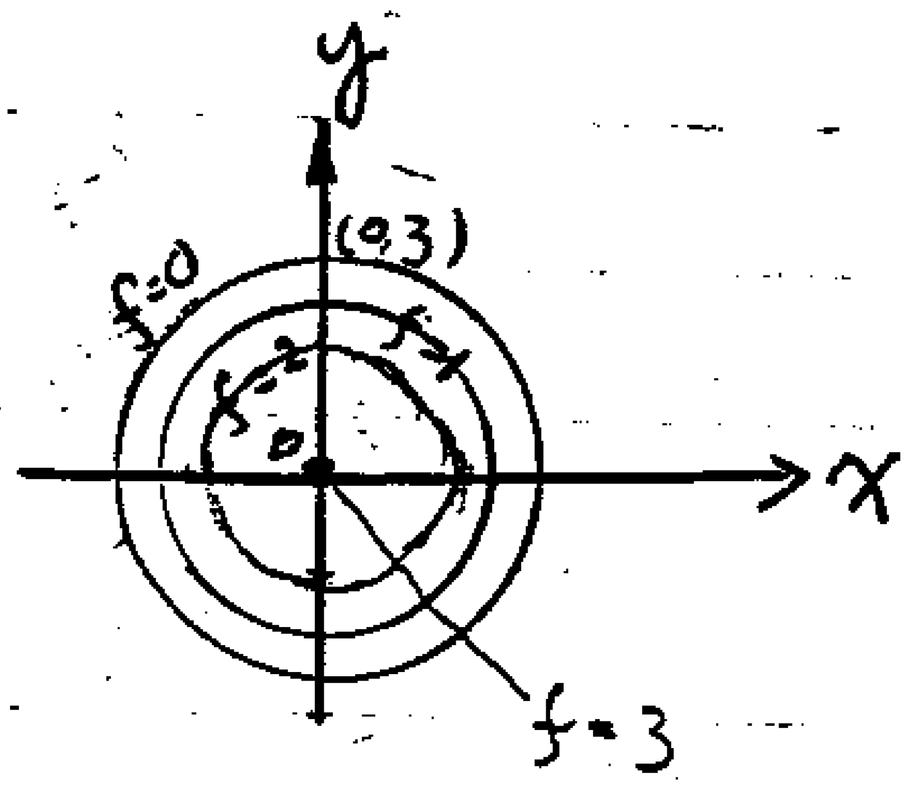
Range $[0, +\infty)$, $f(x,y)$ is continuous in its domain since $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$



(b) The graph $z = f(x,y) = \sqrt{9 - x^2 - y^2}$

The traces in $z = k$ plane $\begin{cases} z = k \\ z = \sqrt{9 - x^2 - y^2} \end{cases}$

(c) $\sqrt{9 - x^2 - y^2} = k$, $k \geq 0$



$k = 0$ $x^2 + y^2 = 3^2$

$k = 1$ $x^2 + y^2 = 8$

$k = 2$ $x^2 + y^2 = 5$

$k = 3$ $x^2 + y^2 = 0$

As k increases, the level curves get sparser.

(d) $\dfrac{\partial f}{\partial x} = -\dfrac{x}{\sqrt{9 - x^2 - y^2}}$, $\dfrac{\partial f}{\partial y} = -\dfrac{y}{\sqrt{9 - x^2 - y^2}}$, $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{-xy}{(9 - x^2 - y^2)^{3/2}}$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{\sqrt{9 - x^2 - y^2}} + \frac{y^2}{(9 - x^2 - y^2)^{3/2}}$$

If both $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are continuous, then $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

(e) $\dfrac{\partial f}{\partial x}(1,1) = -\dfrac{1}{\sqrt{7}}$, $\left.\dfrac{\partial f}{\partial y}\right|_{(1,1)} = -\dfrac{1}{\sqrt{7}}$, $z_0 = f(1,1) = \sqrt{7}$

The eqn. of the tangent plane is

$$z - \sqrt{7} = -\frac{1}{\sqrt{7}}(x - 1) - \frac{1}{\sqrt{7}}(y - 1)$$

(f) $\dfrac{\vec{u}}{|\vec{u}|} = \dfrac{\langle -1, 2 \rangle}{\sqrt{5}} = \left\langle -\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$

$$D_{\vec{u}}f = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|} = \left\langle -\frac{x}{\sqrt{9-x^2-y^2}}, \frac{-y}{\sqrt{9-x^2-y^2}} \right\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$= \frac{x-2y}{\sqrt{5}\sqrt{9-x^2-y^2}}, \quad D_{\vec{u}}f(1,1) = -\frac{1}{\sqrt{35}},$$

$f(x,y)$ decreases along $\vec{u}$.

(8) $\nabla f = \left\langle -\frac{x}{\sqrt{9-x^2-y^2}}, -\frac{y}{\sqrt{9-x^2-y^2}} \right\rangle$

maximum rate of increase $|\nabla f| = \sqrt{\frac{x^2+y^2}{9-x^2-y^2}}$

" " of decline $-|\nabla f| = -\sqrt{\frac{x^2+y^2}{9-x^2-y^2}}$

At $(1,1)$, $|\nabla f| = \sqrt{\frac{2}{7}}$, $-|\nabla f| = -\sqrt{\frac{2}{7}}$.

4. The curvature is the rate of change of the direction with respect to arc-length

$$\kappa = \left| \frac{d\vec{T}(t)}{ds} \right|$$

The circle $x^2+y^2=R^2$ in parametric form in xyz-space can be written as

$$\begin{cases} x = R\cos t \\ y = R\sin t \\ z = 0 \end{cases}$$

$$\vec{r}'(t) = \langle -R\sin t, R\cos t, 0 \rangle$$

$$\vec{r}''(t) = \langle -R\cos t, -R\sin t, 0 \rangle$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R\sin t & R\cos t & 0 \\ -R\cos\theta & -R\sin t & 0 \end{vmatrix} = \vec{k} R^2$$

$$K(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{R^2}{R^3} = \frac{1}{R}$$

The curvature decreases as the circle gets bigger.

5 $\frac{\partial f}{\partial x} = 4x^3 - 4y = 0$ $\quad x^3 - y = 0 \Rightarrow x^9 = y^3$

$\frac{\partial f}{\partial y} = 4y^3 - 4x = 0,$ $\quad y^3 - x = 0$

$x^9 - x = 0.$ $\quad x(x^8 - 1) = 0$ $\quad x(x^4+1)(x^4-1) = 0$

or $x(x^4+1)(x^2+1)(x^2-1) = 0$

$x = 0, \quad x = 1, \quad x = -1$ $\qquad$ Three critical points

$y = x^3, \Rightarrow y = 0, \; y = +1, \; y = -1.$ $\quad (0,0), (1,+1), (-1,-1)$

$f_{xx} = 12x^2, \quad f_{xy} = -4, \quad f_{yy} = 12y^2$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$$

At $(0,0)$, $D = -16$, a sandle point

At $(1,1)$, $D = 144 - 16 > 0$, $f_{xx} > 0$, A local minimum

At $(-1,-1)$ $D = 144 - 16 > 0$, $f_{xx} > 0$, $(-1,-1)$ is also a local minimum.

6. (a) $K(B)$ is bigger because the curve at A is flatter.

(b). $\vec{v} = -\nabla f$ because $\vec{v}$ is orthogonal to the level curves and pointing the direction in which $f(x,y)$ decreases.

(3) Apparently, $f(x,y)$ has a local minimum.

$f(\text{center}) = \text{minimum of } f \approx 972$