Solution to Quiz 2

1. (a) $\vec{r}\,'(t) = \langle -e^{-t}, 2t, 2 \rangle$

(b) $|\vec{r}\,'(t)| = \sqrt{e^{-2t} + 4t^2 + 4}$

$$L = \int_0^s \sqrt{e^{-2t} + 4t^2 + 4}\, dt$$

(c) $\vec{r}(0) = (1, 0, 0)$, $\vec{r}\,'(0) = \langle -1, 0, 2 \rangle$

The Eqn. of the tangent line is

$$\begin{cases} x(t) = 1 - t \\ y(t) = 0 \\ z(t) = 2t \end{cases}$$

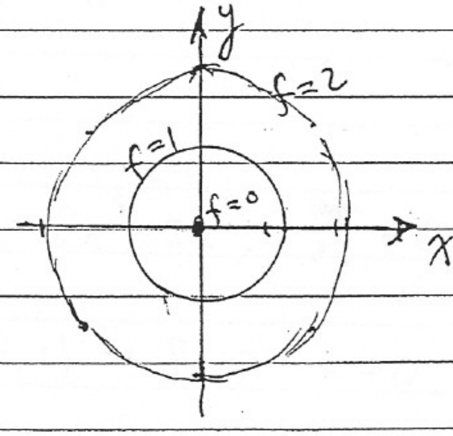
2. (a) Domain all points on x-y plane

(b) range $[0, +\infty)$, $f(x,y)$ is continuous everywhere.

(c) $f = 0 \Rightarrow 0 = x^2 + y^2$

$f = 1 \qquad 1 = x^2 + y^2$

$f = 2 \qquad 4 = x^2 + y^2$



(d) The curvature is $\kappa = 1$ for the level curve $f=1$

The curvature is $\kappa = \frac{1}{2}$ for the level curve $f=2$, or $2^2 = x^2 + y^2$

3. (a) $\dfrac{\partial f}{\partial x} = \cos x\, e^y$, $\dfrac{\partial f}{\partial y} = \sin x\, e^y$, $\dfrac{\partial^2 f}{\partial x \partial y} = \cos x\, e^y$

(b) $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle \overset{\sin}{\cancel{\cos}} x e^y, \cos x\, e^y \rangle$

(c) $f(\frac{\pi}{4}, 0) = \frac{1}{\sqrt{2}}$, The point is $(\frac{\pi}{4}, 0, \frac{1}{\sqrt{2}})$

$\frac{\partial f}{\partial x}(\frac{\pi}{4}, 0) = \frac{1}{\sqrt{2}}$, $\frac{\partial f}{\partial y}(\frac{\pi}{4}, 0) = \frac{1}{\sqrt{2}}$

The eqn. of the tangent plane is

$$z - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) + \frac{1}{\sqrt{2}} y$$

(d) $\frac{\vec{u}}{|\vec{u}|} = \frac{\langle 1, -1 \rangle}{\sqrt{2}} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

$D_{\vec{u}} f(\frac{\pi}{4}, 0) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(-\frac{1}{\sqrt{2}}) = 0$

(e) $-\nabla f$ or $-\frac{\nabla f}{|\nabla f|} = \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$.

The rate is $-|\nabla f| = -1$

4 $\nabla f = \langle 8x^3 - 2x, 6y \rangle$, $\frac{\partial f}{\partial x} = 8x^3 - 2x$, $\frac{\partial f}{\partial y} = 6y$

$\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$. $\frac{\partial f}{\partial x}(1,1) = 8 - 2 = 6 \neq 0$

$\frac{\partial f}{\partial x}(\frac{1}{2}, 0) = 8 \cdot (\frac{1}{2})^3 - 2 \cdot \frac{1}{2} = 0$, $\frac{\partial f}{\partial y}(\frac{1}{2}, 0) = 0$

Therefore $(0,0)$ and $(\frac{1}{2}, 0)$ are critical points

$f_{xx} = 24x^2 - 2$, $f_{xy} = 0$, $f_{yy} = 6$

$f_{xx}(0,0) = -2$, $\quad D(0,0) = \begin{vmatrix} -2 & 0 \\ 0 & 6 \end{vmatrix} = -12 < 0$

$(0,0)$ is a saddle point.

$f_{xx}(\frac{1}{2},0) = 24 \cdot \frac{1}{4} - 2 = 4$, $\quad D(\frac{1}{2},0) = \begin{vmatrix} 4 & 0 \\ 0 & 6 \end{vmatrix} = 24 > 0$.

So $(\frac{1}{2},0)$ is a local minimum.

Extra Credit:

(a) Use $2|xy| \le x^2+y^2$, we have

$$0 \le \left| \frac{x^2y}{x^2+y^2} - 0 \right| \le \frac{2|x|\ |x||y|}{2(x^2+y^2)} \le \frac{2|x|(x^2+y^2)}{2(x^2+y^2)}$$

$$\le 2|x|$$

Since $\lim_{(x,y)\to(0,0)} 2|x| = 0$, from the sandwich technique, we have proved

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0.$$

(b) The equ. of the trace is $\begin{cases} z = \sin\frac{\pi}{4}\, e^y \\ x = \frac{\pi}{4} \end{cases}$

or $\begin{cases} x = \frac{\pi}{4} \\ y = t \\ z = \frac{1}{\sqrt{2}} e^t \end{cases}$, $\quad \begin{cases} x' = 0 \\ y' = 1 \\ z' = \frac{1}{\sqrt{2}} e^t \end{cases}$ $\quad z'(0) = \frac{1}{\sqrt{2}}$

Therefore the line equ is

$$\begin{cases} x = \frac{\pi}{4} \\ y = 0 + t \\ z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} t \end{cases}$$