Finite difference methods, Green functions and error analysis, IB/IIM methods

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Finite difference basics

References
- Some papers
- Matlab Codes: https://zhilin.math.ncsu.edu/TEACHING/MA584/index.html

Course Delivery Method: Classes will be delivered online during the class time using Zoom. The classes will be recorded. The video and notes will be available.

Note that: In case of possible Internet outages, please wait up to for 5 minutes or so to get reconnected.
Differential Equations (ODE/PDEs)
- Initial value problems (IVPs)
- Boundary value problems (BVPs)
- Initial and boundary value problems, have to be time dependent

Classification of ODE/PDEs
- Mathematical: order, linearity (quasi-linear), constant/variable coefficient(s), homogeneity/source terms
- Physical: advection, heat, wave, elliptic equations; Laplace/Poisson equations
- Hyperbolic, elliptic*, parabolic* (compare with quadratic curves)

Well-posedness of ODE/PDES, existence, uniqueness, and sensitivity

Solution techniques
- Analytic: PDE convert to ODE
- Approximate semi-analytic; numerical solutions
FDM for IVP problems, Matlab ODE-Suite

- Canonical form (no space derivative)

\[
\frac{dy}{dt} = f(t, y), \\
y(t_0) = v,
\]

(1)

- Well-posed if \( f \) is Lipschitz continuous.

\[
\frac{dy_1}{dt} = f_1(t, y_1(t), y_2(t), \cdots, y_m(t)), \\
\frac{dy_2}{dt} = f_2(t, y_1(t), y_2(t), \cdots, y_m(t)), \\
\vdots \\
\frac{dy_m}{dt} = f_m(t, y_1(t), y_2(t), \cdots, y_m(t)),
\]

(2)

- High order would be converted to the standard form if possible.
- \texttt{ode23}, \texttt{ode15s}, \texttt{ode45}, etc. Useful for method of lines (MOL)
- FDM, forward/backward/central Euler method, Crank-Nicholson, Runge-Kutta, adaptive time steps etc.
Finite difference methods

- Long history. Simple and intuitive for regular domains (research for arbitrary ones)
- Point-wise discretization, error measure in the infinity norm
- Strong form
- Easier to obtain approximate derivatives with the same order of accuracy as the solution
- Compact and local, fast solvers, structured meshes
- Difficult for arbitrary geometry, smaller group
- Hard to prove the convergence
- Finite volume (FV) methods are special types of FDM.

Finite difference methods vs Finite element methods
Finite element methods, short history (1950-60’s)

- Based on integral forms, testing function spaces and solution spaces. The solution is approximate by simple, piecewise functions
- Weak form, 2nd to first, 4th to second
- Lower order accuracy for derivatives, posterior error analysis
- Solid theoretical foundations based on Sobolev theory, Lax-Milgram, Lax-Milgram-Breeze
- Used to prove wellposedness of models
- Natural for mechanical problems
- Natural for general geometries but need expert knowledge in programming except for structure meshes.
Key: representing derivative(s) using combination of function values on grid points (finite)

- Basic finite difference formulas and **Operator notation**

  Forward: \[ \Delta_+ u(\bar{x}) = \frac{u(\bar{x} + h) - u(\bar{x})}{h} = u'(\bar{x}) + O(h) \]

  Backward: \[ \Delta_- u(\bar{x}) = \frac{u(\bar{x}) - u(\bar{x} - h)}{h} = u'(\bar{x}) + O(h) \]

  Central: \[ \delta u(\bar{x}) = \frac{u(\bar{x} + h) - u(\bar{x} - h)}{2h} = u'(\bar{x}) + O(h^2) \]

- Discretization, discretization error, order of discretization

- **Q:** Can we take \( h \) arbitrarily small?
Finite difference formula for second order derivatives

- **Central**

\[
\delta^2 u(\bar{x}) = \frac{u(\bar{x} - h) - 2u(\bar{x}) + u(\bar{x} + h)}{h^2} = u''(\bar{x}) + O(h^2)
\]

\[
= \Delta_+ \Delta_- u(\bar{x}) + O(h^2)
\]

- **One-sided**

\[
\Delta_+ \Delta_+ u(\bar{x}) = \frac{u(\bar{x}) - 2u(\bar{x} + h) + u(\bar{x} + 2h)}{h^2} = u''(\bar{x}) + O(h)
\]

\[
\Delta_- \Delta_- u(\bar{x}) = \frac{u(\bar{x} - 2h) - 2u(\bar{x} - h) + u(\bar{x})}{h^2} = u''(\bar{x}) + O(h)
\]

- **Q:** Do we have high order FD discretizations (Consider, the number of points, whether it is compact etc.)
Process of a finite difference method, notations

Use the example (when is it wellposed):

\[ u''(x) = f(x), \quad a < x < b, \quad u(a) = u_a, \quad u(b) = u_b, \]

If \( f(x) \in C(a, b) \), then \( u(x) \in C^2(a, b) \) (FEM: if \( f(x) \in L^2(a, b) \), then \( u(x) \in H^2(a, b) \)).

- Generate a grid, e.g. a uniform grid

\[ x_i = a + i \, h, \quad i = 0, 1, \ldots n, \quad h = \frac{(b - a)}{n}, \quad x_0 = a, \quad x_n = b, \]

with \( n \) being a parameter (control accuracy). \( x_i \)'s are called grid points.

- At every grid point where the solution is unknown, substitute the DE with a finite difference approximation

\[ u''(x_i) = \frac{u(x_i - h) - 2u(x_i) + u(x + h)}{h^2} + \frac{u^{(4)}(\xi_i)}{12} h^2 = f(x_i) \]
The $T_i = \frac{u^{(4)}(\xi_i)}{12} h^2$ is called the local truncation error (unknown).

- The finite difference solution (if exists) is defined as the solution of

\[
\frac{u_a - 2U_1 + U_2}{h^2} = f(x_1)
\]

\[
\frac{U_1 - 2U_2 + U_3}{h^2} = f(x_2)
\]

\[
\cdots = \ldots
\]

\[
\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i)
\]

\[
\cdots = \ldots
\]

\[
\frac{U_{n-3} - 2U_{n-2} + U_{n-1}}{h^2} = f(x_{n-2})
\]

\[
\frac{U_{n-2} - 2U_{n-1} + u_b}{h^2} = f(x_{n-1}).
\]

So $U_i \approx u(x_i)$. How large/small is the difference? Need error analysis.
For this ODE, after finite difference, the problem becomes a linear system of equations $A_h U = F$, $A_h$ is called discrete Laplacian,

$$
\begin{bmatrix}
-\frac{2}{h^2} & \frac{1}{h^2} & & & \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & & \\
& \ddots & \ddots & \ddots & \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & & \\
\frac{1}{h^2} & -\frac{2}{h^2} & & & \\
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
\vdots \\
U_{n-2} \\
U_{n-1} \\
\end{bmatrix}
= 
\begin{bmatrix}
f(x_1) - \frac{u_a}{h^2} \\
f(x_2) \\
\vdots \\
f(x_{n-2}) \\
f(x_{n-1}) - \frac{u_b}{h^2} \\
\end{bmatrix}
$$

$A_h$ is tridiagonal, symmetric, $-A_h$ is an SPD, an M-matrix

- The diagonals have one sign, all off-diagonals have opposite sign
- It’s (weakly diagonally) row/column dominant, at least one row is strictly, irreducible
- Fast solver, chase method, $O(5N)$ complexity.

$A_h$ is invertible. The system of the FD equations has a unique solution.
Convergence analysis: Need consistency & stability

**Convergence:** \( \lim_{h \to 0} |u(x_i) - U_i| = 0 \) for all \( i \)'s. Better to use \( \lim_{h \to 0} \| u - u \| = 0 \) in some norm, which one? \( \| \cdot \|_\infty, \| \cdot \|_{L^2} \). **Called global error!**

Theorem: If a FDM is consistent & stable, then it is convergent. (A sufficient condition!)

- Consistency often is easy, \( \lim_{h \to 0} T_i = 0 \). \( \| DE - FD \|_{\Omega_h} \to 0 \).
  
  May have different expressions. Should use the correct one!

- **Stability** is often challenging for both FDM and FEM (sup-inf condition). Not many people pay attention to elliptic problems. Require: \( \| A_h \| \leq C \) so that the global error has the same order as the discretization, no deterioration in accuracy!

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*Finite difference methods, Green functions and error analysis, IB/IIM methods*
An example of consistency but no stability

If we use a consistent first order, one-sided FD formula at $x_1$

$$\frac{U_1 - 2U_2 + U_3}{h^2} = f(x_1)$$

What will happen? We have $T_1 \sim O(h)$ so it’s consistent, but $det(A_1) = 0$ and there is no unique solution to $A_1 U = F$. The BC is not used!

The FD method fails!
Convergence proof and the optimal error estimate

The **global error** vector \( E = u - U \).

\[
A_h u = F + T, \quad A_h U = F \quad \Rightarrow \quad A_h (u - U) = T = -A_h E, \quad E = A_h^{-1} T.
\]

We know \( \| T \|_\infty \leq C h^2 \), but it’s not so easy to estimate \( \| A_h^{-1} T \| \)!

- Use eigenvalues of \( A_h \). (R. LeVeque)
- Use discrete Green functions.
- Use a comparison function. (Morton & Mayers, J. H. Bramble)
Use eigenvalues of $A_h$

Explicit expression for tridiagonal matrices $(\alpha, d, \alpha)$$A = \text{full}(\text{gallery}('\text{tridiag}', n, -1, 2, -1))/h^2$

Eigenvalues and eigenvectors

$$\lambda_j = -2 + 2 \cos \frac{\pi j}{n}, \quad j = 1, 2, \cdots n - 1,$$

$$\chi_k^j = \sin \frac{\pi kj}{n}, \quad k = 1, 2, \cdots n - 1.$$

$$\|A_h\|_2 = \frac{2}{h^2} \left(1 - \cos \frac{\pi \text{int}(n/2)}{n}\right) \sim \frac{2}{h^2}$$

$$\|A_h^{-1}\|_2 = \frac{1}{\min |\lambda_j|} = 2 \left(1 - \cos \frac{\pi}{n}\right) \sim \frac{1}{\pi^2}$$

$$\text{cond}_2(A_h) = \|A_h\|_2 \|A_h^{-1}\|_2 \sim 4n^2$$

Large if $n$ is large!
Error estimate using eigenvalues

Using the inequality $\| A_h^{-1} \|_\infty \leq \sqrt{n - 1} \| A_h^{-1} \|_2$, we have

$$\| E \|_\infty \leq \| A_h^{-1} \|_\infty \| T \|_\infty \leq \sqrt{n - 1} \| A_h^{-1} \|_2 \| T \|_\infty \leq \frac{\sqrt{n - 1}}{\pi^2} Ch^2 \leq \tilde{C} h^{3/2},$$

A decent estimate but not optimal. Expect to have $O(h^2)$!
Consider one error component $E_i$

$$E_i = \sum_{k=1}^{N-1} (A_h^{-1})_{ik} T_k = h \sum_{k=1}^{N-1} (A_h^{-1})_{ik} \frac{1}{h} e_k = h \sum_{k=1}^{N-1} T_k (A_h^{-1})_{ik} \frac{1}{h} e_k$$

**Definition of discrete Green function in 1D:**

$$G(x_i, x_l) = \left( A_h^{-1} e_l \frac{1}{h} \right)_i, \quad G(x_0, x_l) = 0, \quad G(x_n, x_l) = 0. \quad (3)$$

Analogue to a continuous problem

$$g''(x) = \delta(x - x_i), \quad g(a) = 0, \quad g(b) = 0. \quad (4)$$

The solution is:

$$g(x, x_i) = \begin{cases} (b - x_i) x, & a < x < x_i, \\ (b - x) x_i, & x_i < x < b. \end{cases} \quad (5)$$
Discrete Green function II

- \( G(x_i, x_l) = g(x_i, x_l) \), may not be true in 2D.
- Symmetry \( G(x_i, x_l) = G(x_l, x_i) \).
- \( O(1/h) \) at one point \( \implies \) \( O(1) \) globally!
- 2D, \( g = \log r \), 3D, \( g = \frac{1}{r} \), discrete with different BC’s.

\[
E_i = \sum_{k=1}^{N-1} (A_h^{-1})_{ik} T_k = h \sum_{k=1}^{N-1} (A_h^{-1})_{ik} T_k \frac{1}{h} e_k = h \sum_{k=1}^{N-1} T_k (A_h^{-1})_{ik} \frac{1}{h} e_k
\]

\[
|E_i| = \left| h \sum_{k=1}^{N-1} T_k (A_h^{-1})_{ik} \frac{1}{h} e_k \right| \leq Ch^3 \sum_{k=1}^{N-1} G(x_i, x_k) \leq Ch^2 (b - a) \max\{|a|, |b|\}, \quad C = \frac{u_{xxxx}}{12}.
\]
Which ones can you solve, or solve accurately?

- **Pure Neumann BC**
  \[ u''(x) = f(x), \quad a < x < b, \]
  \[ u'(a) = \alpha, \quad u'(b) = u_b, \]

- **Periodic BC**: \[ u''(x) = f(x), \quad a < x < b, \] What does it mean?
  \[ u(a) = u(b), \quad u'(a) = u'(b). \]

- How about this one?
  \[ u''(x) + u(x) = f(x), \quad 0 < x < \pi, \]
  \[ u(a) = 0, \quad u(b) = 0, \]

  Helmholtz equations: \[ u''(x) + k^2 u(x) = f(x), \] generalized
  Helmholtz equations: \[ u''(x) - k^2 u(x) = f(x), \] Good one!

2D, 3D analogues; practical applications. May or may not be solvable.
Ghost point method & analysis

\( u''(x) = f(x), \quad a < x < b, \)

\( u'(a) = \alpha, \quad u(b) = u_b. \)

It’s wellposed. **Method 1:**

\[
\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f_i, \quad i = 1, 2, \ldots, n - 1,
\]

\[
\frac{U_1 - U_0}{h} = \alpha \quad \text{or} \quad \frac{-U_0 + U_1}{h^2} = \frac{\alpha}{h}.
\]

\[
\begin{bmatrix}
-\frac{1}{h^2} & \frac{1}{h^2} \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\text{...} & \text{...} & \text{...} \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\end{bmatrix}
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_{n-2} \\
U_{n-1}
\end{bmatrix}
= \begin{bmatrix}
\frac{\alpha}{h} \\
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_{n-2}) \\
\frac{f(x_{n-1}) - u_b}{h^2}
\end{bmatrix}
\]
First order discretization of BC, second order discretization of DE at interior points, expect global error $\|E\|_\infty \sim O(h)$. There are $n$ equations, and $n$ unknowns, matched. $A_h$ is an M-matrix, invertible. Can you prove the convergence? What’s the local truncation error? $O(h^2)$ interior, $O(1)$ at $x_0$, why? $u''(x_0) - u'(x_0) = O(1)$!
Idea: Use second order discretization to the Neumann BC.

**Method 2:** One-sided 2nd order finite difference formula:

\[- \frac{U_2 + 4U_1 - 3U_0}{h} = \alpha\]

Not recommend because (1), the matrix-structure will be changed; (2) hard to prove the stability and convergence.

**Method 3:** The ghost point method. Extend the soln. to \(x_{-1} = x_0 - h\) (outside, ghost point).

- Use ghost point \(x_{-1}\) and ghost value \(U_{-1}\). Second order discretization for the DE and BC. Add

\[ \frac{U_{-1} - 2U_0 + U_1}{h^2} = f_0, \quad \frac{U_1 - U_{-1}}{2h} = \alpha, \]

- There are \(n + 1\) equations, and \(n + 1\) unknowns, matched.
- Eliminate \(U_{-1}\). \(\frac{-2U_0 + 2U_1}{h^2} = f_0 + \frac{2\alpha}{h}\).
The ghost point method II

\[
\begin{bmatrix}
-\frac{2}{h^2} & \frac{2}{h^2} \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\vdots & \ddots & \ddots & \ddots \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
\frac{1}{h^2} & -\frac{2}{h^2}
\end{bmatrix}
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_{n-2} \\
U_{n-1}
\end{bmatrix}
= 
\begin{bmatrix}
f_0 + \frac{2\alpha}{h} \\
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_{n-2}) \\
f(x_{n-1}) - \frac{u_b}{h^2}
\end{bmatrix}
\]

- \( A_h \in R^{n \times n} \) is an M-matrix. The same structure!
- Second order accurate \( \| E \|_\infty \sim O(h^2) \).
- Use the discrete Green function with homogeneous Neumann BC. Project!
- Exercise: Generalize the ghost point method to Robin (mixed) BC and analyze.
What’s the local truncation error at $x_0$? Why is it $O(h)$?

$$T_i = \frac{-2u(x_i) + 2u(x_1)}{h^2} - \frac{2\alpha h}{h^2} - f_0$$

$$= \frac{u(x_{-1}) - 2u(x_i) + u(x_1) + O(h^3)}{h^2} - f_0$$

$$= O(h) + O(h^2)$$

since $u(x_1) = u(x_{-1}) + 2h\alpha + O(h^3)$.

**Rule of thumb:** *In general, the local truncation error can be one order lower than that in the interior without affecting global accuracy.*
Conserve FD schemes for 1D self-adjoint S-L problem

\[(p(x)u'(x))' - q(x)u(x) = f(x), \quad a < x < b,\]
\[u(a) = u_a, \quad u(b) = u_b, \quad \text{or other BC.}\]

- If \(p(x) \in C^1(a, b), \ q(x) \in C^0(a, b), \ f(x) \in C^0(a, b), \ q(x) \geq 0\) and \(p(x) \geq p_0 > 0\), then it’s wellposed \(u(x) \in C^2(a, b)\).
- **DO NOT** use \(pu'' + p'u' - qu = f\), why?
- Use a staggered, (or MAC?) approach! First discretize one derivative at \(x_{i+\frac{1}{2}} = x_i + h/2\).

\[
p_{i+\frac{1}{2}} \frac{u'(x_{i+\frac{1}{2}}) - p_{i-\frac{1}{2}} u'(x_{i-\frac{1}{2}})}{h} - q_i u(x_i) = f(x_i) + E_{i}^1, \]
\[
p_{i+\frac{1}{2}} \frac{u(x_{i+1}) - u(x_i)}{h} - p_{i-\frac{1}{2}} \frac{u(x_i) - u(x_{i-1})}{h} - q_i u(x_i) = f(x_i) + E_{i}^1 + E_{i}^2, \]
The matrix-vector form

\[
\frac{p_{i+\frac{1}{2}}}{h^2} U_{i+1} - \left(\frac{p_{i+\frac{1}{2}}}{h^2} + \frac{p_{i-\frac{1}{2}}}{h^2}\right) U_i + \frac{p_{i-\frac{1}{2}}}{h^2} U_{i-1} - q_i U_i = f_i,
\]

for \(i = 1, 2, \ldots n - 1\). In a matrix-vector form, this linear system can be written as \(A_h U = F\), where

\[
A_h = \begin{bmatrix}
- \frac{p_{1/2} + p_{3/2}}{h^2} - q_1 & \frac{p_{3/2}}{h^2} & & & \\
\frac{p_{3/2}}{h^2} & - \frac{p_{3/2} + p_{5/2}}{h^2} - q_2 & \frac{p_{5/2}}{h^2} & & \\
& \ddots & \ddots & \ddots & \\
& & & \frac{p_{n-3/2}}{h^2} & - \frac{p_{n-3/2} + p_{n-1/2}}{h^2} - q_{n-1}
\end{bmatrix},
\]

\[
U = \begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
\vdots \\
U_{n-2} \\
U_{n-1}
\end{bmatrix}, \quad F = \begin{bmatrix}
f(x_1) - \frac{p_{1/2} u_a}{h^2} \\
f(x_2) \\
f(x_3) \\
\vdots \\
f(x_{n-2}) \\
f(x_{n-1}) - \frac{p_{n-1/2} u_b}{h^2}
\end{bmatrix}.
\]
- $A_h$ is still tridiagonal.
- $A_h$ is an M-matrix if the regularity conditions are satisfied.
- The discretization and global solution are both second order accurate! $T_i \sim O(h^2)$, $\|E\|_\infty \sim O(h^2)$. 
Not self-adjoint, diffusion and advection equation

- A boundary layer problem:
  \[ \epsilon u'' - u' = -1, \quad 0 < x < 1, \]
  \[ u(0) = 1, \quad u(1) = 3. \]

- More general form
  \[ (p(x)u'(x))' + b(x)u'(x) - q(x)u(x) = f(x), \quad a < x < b, \]
  \[ u(a) = u_a, \quad u(b) = u_b, \quad \text{or other BC}. \]

- Central schemes if the advection is small \( b \leq \frac{C}{h} \).
- Upwinding, first order
- Integrating factor + scaling \( \Rightarrow \) Change the problem to a self-adjoint? How?
Integrating factor + scaling strategy

- Idea: multiplying an integrating factor to eliminate the advection term.

\[ \mu \left\{ (pu')' + bu' - qu \right\} = \mu f \]

\[ \Rightarrow (\mu pu')' - \mu' pu' + \mu bu' - q\mu u = \mu f \]

\[ \Rightarrow \mu(x) = \int e^{b(x)/p(x)} \, dx \]

- New ODE is a diffusion equation. Integral operation is a good operator. Let \( \bar{p} \) be the average of \( p \) in \( (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \).

\[ (\mu pu')' - q\mu u = \mu f, \quad \text{scaling} \]

\[ \frac{1}{\bar{p}} \left\{ \frac{p_{i+\frac{1}{2}} U_{i+1} - \left(p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}}\right) U_i + p_{i-\frac{1}{2}} U_{i-1}}{h^2} - q_i U_i = f_i \right\} \]
Use known exact solution and check how the error changes called grid refinement analysis

\[ n = 40, \| E_{40} \|, \quad n = 80, \| E_{80} \|, \quad \frac{\| E_{40} \|}{\| E_{80} \|}, \quad \cdots, \]

ratios \[ \frac{\| E_{N} \|}{\| E_{2N} \|}, \quad \text{rate} \quad p = \left| \log \left( \frac{\| E_{N} \|}{\| E_{2N} \|} \right) \right| \]

How to choose exact solutions? Set \( u \) first, find \( f \) and BCs, called manufactured solution. Not always easy!

First order method, ratio \( \approx 1 \), oder \( \approx 1 \); Second order method, ratio \( \approx 4 \), oder \( \approx 2 \).
If problem is too complicated, then we can use the solution obtained from a finest grid, say, $n = 1024$, but the ratios and rates will be different.

Wrongly used in the literature, discovered by Zhilin Li

\[ u_h = u_e + Ch^p + \cdots \]

\[ u_{h^*} = u_e + Ch_{h^*}^p + \cdots \]

\[ u_h - u_{h^*} \approx C (h^p - h_{h^*}^p), \]

\[ u_{h/2} - u_{h^*} \approx C ((h/2)^p - h_{h^*}^p) \]

\[ \frac{\tilde{u}(h) - \tilde{u}(h^*)}{\tilde{u}(\frac{h}{2}) - \tilde{u}(h^*)} = \frac{2^p (1 - 2^{-kp})}{1 - 2^p(1-k)} \]

\[ \frac{7}{3} \approx 2.333, \quad \frac{15}{7} \approx 2.1429, \quad \frac{31}{15} \approx 2.067, \quad \cdots \]

\[ \frac{63}{15} = 4.2, \quad \frac{255}{63} \approx 4.0476, \quad \frac{1023}{255} \approx 4.0118, \quad \cdots \]
A fourth order compact scheme for 1D Poisson equation

\[ u''(x) = f(x), \quad a < x < b, \]

\[ u(a) = \alpha, \quad u(b) = u_b. \]

- It's possible to use a 5-point stencil 4th order scheme, not recommended, why?
  Not compact, does not work near BC, matrix structure changed (not an M-matrix), hard to discuss the stability.
- Use finite difference operator to derive high order compact schemes.

\[
\delta_{xx}^2 u = \frac{d^2 u}{d x^2} + \frac{h^2}{12} \frac{d^4 u}{d x^4} + O(h^4)
\]

\[
= \left( 1 + \frac{h^2}{12} \frac{d^2}{d x^2} \right) \frac{d^2}{d x^2} u + O(h^4)
\]

\[
= \left( 1 + \frac{h^2}{12} \delta_{xx}^2 \right) \frac{d^2}{d x^2} u + O(h^4),
\]
A fourth order compact scheme for 1D Poisson equation

Solve for $\frac{d^2 u}{dx^2}$

$$\frac{d^2}{dx^2} u = \left(1 + \frac{h^2}{12} \delta_{xx}^2 \right)^{-1} \delta_{xx}^2 u + O(h^4)$$

The fourth order compact scheme is

$$\left(1 + \frac{h^2}{12} \delta_{xx}^2 \right)^{-1} \delta_{xx}^2 U_i = f_i$$

$$\delta_{xx}^2 U_i = \left(1 + \frac{h^2}{12} \delta_{xx}^2 \right) f_i$$

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i) + \frac{1}{12} \left( f_{i-1} - 2f_i + f_{i+1} \right)$$

Motivations (smaller matrices, oscillatory solutions), other methods, Richardson extrapolation.