E1.1(a) : $y(x) = Ce^{-x} + 1$. The first term is the solution to $y' + y = 0$, the second term can be spotted easily or obtained using the undetermined coefficient method.

E1.1(b) : $y(x) = Ce^{-x/2}$ by re-writing the DE as $y' + y/2 = 0$.

E1.1(c) : $y(x) = -\frac{x^2}{4} + 4$.

E1.1(d) : $y(x) = Ce^{2x} - \frac{2}{5}\sin x - \frac{1}{5}\cos x$. The first term is the solution to $y' - 2y = 0$, the second term can be solved using the undetermined coefficient method.

E1.1(e) : The equation is called the Euler equation. We can use the solution for the Euler equation. We can also use Maple to solve it easily using command "ode := diff(y(x), x) = -x*y(x) + x", followed by "dsolve(ode)" . The solution is $y(x) = 1 + Ce^{-x^2/2}$.

E1.1(f) : The solution is $y(x) = Cx$. Re-write the equation as $\frac{dy}{y} = \frac{dy=x}{x}$ and integrate on both sides to get $\log|y| = \log|x| + C_1$, or $y = Cx$ after taking exponential on both sides, where $C = e^{C_1}$.

E1.2 : Reference: https://en.wikipedia.org/wiki/Hyperbolic_functions. They have many similar properties and trigonometric functions.

(a): We have $\sinh'(x) = \cosh(x)$, $\sinh''(x) = \cosh'(x) = \sinh(x)$, so we have $\sinh''(x) - \sinh(x) = 0$, thatis, the DE is satisfied. Similarly, $\cosh'(x) = \sinh(x)$, $\cosh''(x) = \sinh'(x) = \cosh(x)$ so we have $\cosh''(x) = \cosh(x)$, thatis, the DE is satisfied.

(b): If we add/subtract $\sinh(x)$ and $\cosh(x)$, and then divide by number 2, we have

$$e^x = \sinh(x) + \cosh(x), \qquad e^{-x} = \cosh(x) - \sinh(x).$$

(c) $W(\sinh(x), \cosh(x)) = det\begin{pmatrix} \sinh(x) & \sinh'(x) \\ \cosh(x) & \cosh'(x) \end{pmatrix} = det\begin{pmatrix} \sinh(x) & \cosh(x) \\ \cosh(x) & \sinh(x) \end{pmatrix} = \sinh^2(x) - \cosh^2(x) = -1 \neq 0$. Thus, they are two independent solution to the ODE and another other solutions can be expressed as a linear combination of the two.

E1.3 : (a), $u(x,t) = F(t)$; (b), $u(x,t) = G(x)$; (c), $u(x,t) = tf(x) + G(t)$; (d), $u(x,t) = F(x) + G(t)$.

E1.4(a) : $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial^2 u}{\partial x^2} = 2$. $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial^2 u}{\partial y^2} = 2$. Thus, the left hand side of the PDE is $LHS = 2 + 2 = 4 = RHS$.

E1.4(b) : By hand or using Maple, we get: $\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{x^2+y^2} - \frac{x^2}{x^2+y^2}$. $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$, $\frac{\partial^2 u}{\partial y^2} = \frac{1}{x^2+y^2} - \frac{y^2}{x^2+y^2}$. Thus, the left hand side of the PDE is $LHS = 0 = RHS$.

E1.5 : Using Maple commands: `u:=e^(-2^2/4*k*t)/sqrt(4*k*t*pi)` . Then we type: `diff(u,t)-k*diff(u,x,x)` and `simplify(%)`. Note the fact that $\log e = 1$ to get $LHS = RHS$.

Chapter 2

E2.1 : Integrate with respect to x to get $u(x,y) = G(y)$, integrate with resect to x to get $u(x,y) = f(x) + g(y)$, where $f(x)$ and $g(x)$ are arbitrary functions that have up to second order continuous derivatives. Note that the solution includes the case $u(x,y) = F(x) + G(y) + C$.

E2.2 : A second order, linear, constant, non-homogeneous PDE. If $A = 0$, it is a Poisson equation, an elliptic PDE. If $A \neq 0$, then the PDE is a parabolic PDE.

E2.3 : (a) $u(x,t) = f(x - 3t/2)$, $u(x,t) = \sin(x - 3t/2)$. $u(x,t) = e^{-(x-3t/2)^2}$.
(b): $u(x,t) = f(x - bt/a)e^{t/a}$. Note that we have $U' - U/a = 0$ after changing variables. $f(x) = u_0(x)$, $u(x,t) = \sin(x - bt/a)e^{t/a}$; $u(x,t) = e^{-(x-bt/a)^2} e^{t/a}$.

E2.4 : (a): $u_t + \sin t u_x = 0$. Solution: $dx/dt = \sin t$, $x = -\cos t + C$, the solution is $u(x,t) = f(C) = f(x + \cos t)$.
(b) : $e^{x^2} u_x + x u_y = 0$. Solution: $dy/dx = e^{-x^2} x$. $y = -e^{-x^2}/2 + C$. $u(x,y) = f(C) = f(y + e^{-x^2}/2)$.

E2.5 : Find the solution to the following transport equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x} = 0, \qquad 0 < x < 1, \quad t > 0, \\ &u(x,0) = e^{-x}, \qquad 0 < x < 1, \\ &u(0,t) = t^2, \qquad 0 < t. \end{aligned}$$

Solution: $u(x,t) = e^{-(x-t/2)}$ if $x > t/2$; $u(x,t) = (t - 2x)^2$ if $x < t/2$.

E2.7 (a): Use the standard transform for 1D advection equations, $\xi = x - at$, $\eta = t$, we get $U_\eta = e^{2(\xi + a\eta)} = e^{2\xi} e^{2\eta}$. Solve the ODE with respect to η, we get $U(\xi,\eta) = e^{2\xi} e^{2\eta}/2 + F(\xi)$. We then plug back the original variables to get the solution $u(x,t) = \frac{1}{2} e^{2(x-at)} e^{2t} + F(x - at)$.

(b): We should use the method of characteristics by setting $\frac{dx}{dt} = x$ or $\frac{dx}{x} = dt$. The solution to the ODE is $\log|x| = t + C$, or $xe^{-t} = C$. Thus, the solution is $u(x,t) = f(x^{-t})$.

(c): We should use the method of characteristics by setting $\frac{dx}{dt} = t$. The solution to the ODE is $x = t^2/2 + C$. Thus, the solution is $u(x,t) = f(x - t^2/2)$.

(d): Use the standard transform for 1D advection equations, $\xi = x - 3t$, $\eta = t$, we get $U_\eta = U + \xi\eta$. Solve the ODE with respect to η, we get $U(\xi,\eta) = -\xi\eta - \xi + e^\eta F(\xi)$. We then plug back the original variables to get the solution $u(x,t) = -(x - 3t)t - (x - 3t) + e^t F(x - 3t)$.

Chapter 3

E3.1 (A): $u(x,t) = F(x - ct) + G(x + ct)$.
(a) $u(x,t) = \frac{1}{2}(\sin(\pi x - t) + \sin(\pi x + t))$
(b) $u(x,t) = \frac{1}{2}(\sin(\pi(x-t)) + 3\sin(2\pi(x-t)) + \sin(\pi(x+t)) + 3\sin(2\pi(x+t))) - \frac{1}{2\pi}(\cos(x+t) - \cos(x-t))$
(c): See the Maple file.

E3.2 (a) $u(x,t) = \sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}$.
(c) $u(x,t) = \frac{L}{12\pi c}\sin\frac{3\pi x}{L}\sin\frac{3\pi ct}{L} - \frac{L}{60\pi c}\sin\frac{6\pi x}{L}\sin\frac{6\pi ct}{L}$
(d) $u(x,t) = \frac{1}{4}\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L} + \frac{1}{10}\sin\frac{6\pi x}{L}\cos\frac{6\pi ct}{L} + \frac{L}{12c}\sin\frac{3\pi x}{L}\sin\frac{3\pi ct}{L} - \frac{2L}{35\pi c}\sin\frac{7\pi x}{L}\sin\frac{7\pi ct}{L}$.

E3.3 (a)

1. $u(x,t) = \sin(6\pi x)\cos(12\pi t)$.

   (b) $u(x,t) = \frac{1}{6\pi}\sin(3\pi x)\sin(6\pi t)$.

   (e) $u(x,t) = \sin(6\pi x)\cos(12\pi t) - 7\sin(24\pi x)\cos(48\pi t) + \frac{1}{6\pi}\sin(3\pi x)\sin(6\pi t)$.

Chapter 4

E4.1 $n = 1, 2, \cdots$.

E4.2 (b) $\|1\| = \sqrt{2p}$, for others the norm is $\sqrt{p}$.

(c) $a_0 = \frac{1}{2p}\int_{-p}^{p}|x|dx = \frac{1}{p}\int_0^p xdx = \frac{1}{2p}$. For other n, $a_n = \frac{1}{p}\int_{-p}^{p}|x|\cos\frac{n\pi x}{p}dx = \frac{2}{p}\int_0^p |x|\cos\frac{n\pi x}{p}dx = \frac{2p}{(n\pi)^2}(1-(-1)^n)$. Thus, the expansion is

$$|x| = \frac{1}{2p} + \sum_{n=1}^{\infty}\frac{2p}{(n\pi)^2}(1-(-1)^n)\cos\frac{n\pi x}{p}, \qquad x \in (-p, p).$$

The series is convergent in $[-p, p]$.

E4.3 Determine the constants a and b so that the functions 1, x, and $a+bx+x^2$ are orthogonal on $(-1,\ 1)$.

$a = -1/3$, which can be obtained from $\int_{-1}^{1}(a+bx+x^2)dx = \int_{-1}^{1}(a+x^2)dx = 2a + 2/3 = 0$. From $\int_{-1}^{1}x(a+bx+x^2)dx = 2b/3 = 0$, we get $b = 0$.

E4.5. (a) $\lambda_n = (2n+1)^2 \quad y_n(x) = \sin(2n+1)x, \quad n = 0, 1, 2, \cdots$; (b) No; (c) Zero since the eigenfunctions are orthogonal.

E4.6. It does not make sense to expand in $(-1, 1)$ since the set is not an orthogonal one. It only make sense to expand in $(-\pi, \pi)$ as a classical Fourier series. Since $f(x) = x^2$ is an even function, the coefficient of $b_n = 0$ and we have only $\cos nx$ terms. Thus the cosine expansion is the same as the Fourier series.

See Maple file fourier_x_squared;

$$x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos(nx).$$

If we expand x^2 in terms of $\sin(nx)$, all the coefficients are zero. So it is not very useful, or does not converge!

E4.7. We need to change to the standard Sturm-Liouville form (called seld-adjoint) to judge whether the eigenvalues are regular or singular. (a), regular; (b), $(xy')' + \lambda y = 0$, regular; (c), $(x^2y')' + x\lambda y = 0$ with $w(x) = x$, regular. (d), $\left(\frac{y'}{x}\right)' + \lambda\frac{1}{x}y = 0$ with $w(x) = 1/x$, singular at $x = 0$. (e), $((1-x^2)y')' + (1+\lambda x)y = 0$, singular at $x = \pm 1$.

E4.8. (a) $\lambda_n = (\frac{n}{4})^2, \quad y_n(x) = \sin\frac{nx}{4}$, $n = 1, 2, \cdots$;

(b) $\lambda_n = (4n)^2, \quad y_n(x) = \sin(4nx)$, $n = 1, 2, \cdots$;

(c) $\lambda_n = (\frac{\frac{1}{2}+n}{4})^2, \quad y_n(x) = \cos\frac{\frac{1}{2}+n}{4}x$, $n = 0, 1, 2, \cdots$;

(d) $\lambda_n = (\frac{\frac{1}{2}+n}{4})^2, \quad y_n(x) = \sin\frac{\frac{1}{2}+n}{4}x$, $n = 0, 1, 2, \cdots$;

(e) $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$. From $y(0) + y'(0) = 0$, we get $C_1 - C_2\sqrt{\lambda} = 0$, or $C_1 = C_2\sqrt{\lambda}$. From $y(4\pi) = 0$ we get $C_2(\cos 4\pi\sqrt{\lambda} - \sqrt{\lambda}\sin 4\pi\sqrt{\lambda}) = 0$, or $\sqrt{\lambda} = -\tan 4\pi\sqrt{\lambda}$. The eigenfunctions are $y_n(x) = \cos\sqrt{\lambda_n}x - \sqrt{\lambda_n}\sin\sqrt{\lambda_n}x$.

(f) Similarly the eigenvalues satisfy the equation: $\sqrt{\lambda_n} = \tan 4\pi\sqrt{\lambda_n}$. The eigenfunctions are $y_n(x) = \sin\sqrt{\lambda_n}x$.

E4.9. $\lambda_n = \left(\frac{(\frac{1}{2}+n)\pi}{p}\right)^2, \quad y_n(x) = \cos(\frac{(\frac{1}{2}+n)\pi}{p})x$, $n = 0, 1, 2, \cdots$.

(a): No. (b): Infinite. (c): Zero.

E4.10. (a),

$$u(x,t) = \frac{1}{2}\left((x-2t)e^{-(x-2t)} + (x+2t)e^{-(x+2t)}\right) + \frac{1}{4}\int_{x-2t}^{x+2t} e^{-s}ds.$$

(b),

$$u(x,t) = \sin(4\pi x)\cos(8\pi t) + \frac{1}{8\pi}\sin(4\pi x)\sin(8\pi t).$$

E4.13. Note the different boundary condition $\frac{\partial u}{\partial x}(L,t) = 0$. When we use the method of separation of variables, we have two equations

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < L, \quad X(0) = 0, \quad X'(L) = 0,$$

$$T'(t) + c^2\lambda T(t) = 0.$$

From the first equation, we get $\lambda_n = \left(\frac{(\frac{1}{2}+n)\pi}{L}\right)^2$, $X_n(x) = \sin\frac{(\frac{1}{2}+n)\pi x}{L}$, $n = 0, 1, \cdots$. We then solve for $T(t)$ to get $T(t) = b_n e^{-c^2(\frac{1}{2}+n)^2 t}$. Thus, the solution has the following form:

$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{-c^2(\frac{1}{2}+n)^2 t}\sin\frac{(\frac{1}{2}+n)\pi x}{L}.$$

The coefficient is determined from the initial condition $u(x,0) = f(x)$.

$$b_n = \frac{2}{L}\int_0^2 f(x)\sin\frac{(\frac{1}{2}+n)\pi x}{L}dx = \int_0^2 f(x)\sin\frac{(\frac{1}{2}+n)\pi x}{2}dx.$$

Chapter 5

E5.1. Find the period of the following functions. $f(x) = 5$, $f(x) = \cos x$, $f(x) = \cos(\pi x)$, $f(x) = \cos(px)$, $f(x) = \cos^2 x$, $f(x) = \cos x \sin x$, $f(x) = \cos(x/2) + 3\sin(2x)$, $f(x) = \tan(mx) + e^{\sin(2x)}$, and $\left\{\cos\frac{n\pi x}{L}\right\}_{n=0}^{\infty}$.

The periods are: 0, 2π, 2, $\frac{2\pi}{p}$; π, since $\cos^2 x = \frac{1}{2}(1+\cos(2x)$, π, since $\cos x\sin x = \frac{1}{2}\sin(2x)$, 4π, $\max\{\frac{\pi}{m}, \pi\}$, $2L$.

E5.5. (a): It is itself. In this example, $b_n = 0$ since $\cos(2x)$ is even. Try to match one of n's in the expansion $\cos(2x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, obviously $n = 2$, with $a_2 = 1$ and other $a_n = 0$.

(b): Need full expansion with $L = 1$, $\cos(2x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$, where $a_0 = \frac{1}{2}\int_{-1}^{1} \cos(2x)dx = \int_0^1 \cos(2x)dx$, $a_n = \int_{-1}^{1} \cos(2x)\cos n\pi x dx = 2\int_0^1 \cos(2x)\cos n\pi x dx$.

(c): Need full expansion with $L = 1$, $\cos(2x) \sim \sum_{n=1}^{\infty} b_n \sin n\pi x$, with $b_n = 2\int_0^1 \cos(2x)\sin n\pi x dx$.

(d): It's the same as (b) but the expansion is only valid in $(0, 1)$.

(e): It's itself, same as (a).

E5.11. (a): Since $\left|\frac{\cos nx}{n^2} + \frac{\sin nx}{n^3}\right| \le \left(\frac{1}{n^2} + \frac{1}{n^3}\right)$, and $\sum_{n=1}^{\infty}\left(\frac{1}{n^2} + \frac{1}{n^3}\right)$ converges, the series is uniformly convergent.

(b): Since $\left|\frac{x^n}{n!}\right| \le \frac{10^n}{n!}$ and $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges, the series is uniformly convergent.

(c): Since $\left|\frac{\cos(x/n)}{n}\right| \le \frac{1}{n}$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, we have no conclusion about the uniform convergence.

Chapter 6

E6.2. It is a heat equation so the series solution should look like

$$u(x,t) = \sum_{n=1 \text{ or } 0}^{\infty} a_n X_n\left(\frac{\alpha_n \pi x}{L}\right) e^{-\left(\frac{\alpha_n \pi c}{L}\right)^2 t}.$$

(a): Neumann-Dirichlet, $X_n\left(\frac{\alpha_n \pi x}{L}\right) = \cos(\frac{1}{2} + n)x$, $\frac{\alpha_n \pi}{L} = (\frac{1}{2} + n)$, $a_n = \frac{2}{\pi}\int_0^{\pi} 78\cos(\frac{1}{2} + n)x dx$.

(b): Dirichlet-Dirichlet and a normal mode with $n = 10$, so $u(x,t) = 30\sin(10x)e^{-10^2 3^2 t}$.

(c): Dirichlet-Neumann, $X_n\left(\frac{\alpha_n \pi x}{L}\right) = \sin(\frac{1}{2} + n)x$, $\frac{\alpha_n \pi}{L} = (\frac{1}{2} + n)$,
$a_n = \frac{2}{\pi}\left(\int_0^{\pi/2} 33x\ \sin(\frac{1}{2} + n)x dx + \int_{\pi/2}^{\pi} 33(\pi - x)\ \sin(\frac{1}{2} + n)x dx\right)$.

E6.3. The ODE for the SSS is$0 = c^2\frac{d^2u_s}{dx^2}$ or $u_s''(x) = 0$. So the general solution is $u_s(x) = C_1 + C_2 x$.

(a): The BC's are $u_s(0) = 0$ and $u_s(1) = 0$. We get $u_s(x) = 100x$.

(b): The BC's are $u_s(0) = 0$ and $u_s(\pi) = 0$. We get $u_s(x) = 100$.

E6.8. Given

$$\frac{\partial u}{\partial t} = u + \frac{\partial^2 u}{\partial x^2}.$$

(a) Classify the following PDE.

(b) Solve the boundary value problem of the PDE on the domain $0 < x < \pi$ with

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) = 0, \quad u(x,0) = f(x).$$

(c) Find the steady state solution if it exists.

Solution:

(a): The PDE is a second order, linear, constant coefficient, homogeneous PDE, and parabolic.

(b): The $X(x)$ equation is as usual, $X'' + \lambda X = 0$ with $X(0) = 0$ and $X'(\pi) = 0$. It's Dirichlet-Neumann, so $X_n\left(\frac{\alpha_n \pi x}{L}\right) = \sin(\frac{1}{2}+n)x$, $\frac{\alpha_n \pi}{L} = (\frac{1}{2}+n)$. The $T(t)$ equation is $T' - T - \lambda T + \lambda T = 0$ from $\frac{T' - T}{T} = \lambda$. We get $T_n(t) = e^{(1-(\frac{1}{2}+n)^2)t}$. The series solution is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \, \sin(\frac{1}{2}+n)x \, e^{(1-(\frac{1}{2}+n)^2)t}$$

with $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(\frac{1}{2}+n)x\,dx$, $n = 0, 1, \cdots$.

(c): $0 = u + \frac{d^2 u_s}{dx^2}$, $u_s(0) = 0$ and $\frac{du_s}{dx}(\pi) = 0$.
The solution is $u_s(x) = C_1 \cos x + C_2 \sin x$. From the two boundary condition, we get $u_s(x) = C \sin x$, $0 < x < \pi$.

E6.9. The SSS is the solution to the following:

$$0 = \frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2}, \quad 0 < x < 2, \quad 0 < y < 1,$$

$$u_s(x,0) = 0, \quad \frac{\partial u_s}{\partial y}(x,1) = 0, \quad u_s(0,y) = 0, \quad u_s(2,y) = \sin y.$$

In this problem, if we use $u_s(x,y) = X(x)Y(y)$, then we have $Y(0) = 0$ and $Y'(1) = 0$, and $X(0) = 0$. We have two homogeneous BC's for $Y(y)$ and only one homogeneous BC for $X(x)$. So we should solve the S-L eigenvalue problem for $Y(y)$ which is a Dirichlet-Neumann type with $L = 1$. We have $Y_n(y) = \sin(\frac{1}{2}+n)\pi y$. Then from $X'' - \lambda_n X = 0$ we get $X_n(x) = A_n \cosh(\frac{1}{2}+n)\pi x + B_n \sinh(\frac{1}{2}+n)\pi x$. Since $X(0) = 0$, we have $A_n = 0$, so the series solution is

$$u_s(x,y) = \sum_{n=0}^{\infty} B_n \, \sin(\frac{1}{2}+n)\pi y \, \sinh(\frac{1}{2}+n)\pi x.$$

Finally, using $u(2,y) = \sin y$, we get $B_n = \frac{2}{\sinh(1+n)}\int_0^1 \sin y \, \sin(\frac{1}{2}+n)y\,dy$.

E6.10. The SSS is the solution to the following:

$$0 = \Delta u_s, \qquad x^2 + y^2 < 3,$$

(a): $u_s(3,\theta) = 100$, **Hint: a normal mode.**

(b): $u_s(3,\theta) = \begin{cases} 1 & \text{if } 0 < \theta \le \pi, \\ 0 & \pi < x < 2\pi. \end{cases}$

The solution to (a) is simply $u_s(r,\theta) = 100$. For (b), we can use the formula (6.60) in the book,

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(\frac{r}{3}\right)^n (a_n \cos n\theta + b_n \sin n\theta).$$

which is the Fourier series expansion of $f(\theta)$. The coefficients are

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} f(\theta)d\theta = \frac{1}{2}, \qquad a_n = \frac{1}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta = 0,$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta = -\frac{1}{n\pi}\cos n\theta\Big|_0^{\pi} = \frac{1-\cos n\pi}{n\pi}, \qquad n = 1, 2, \cdots.$$